# On the Role of Pressure in Theory of the Navier–Stokes and MHD Equations I

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# 1. The Navier–Stokes equations and where they come from

**Beginnings of theoretical or mathematical considerations in mechanics of fluids:**

- Aristoteles (384–322 B.C.)
- Archimedes (287–212 B.C.)
- B. Pascal (1623–1662)

⋮

**Important milestone: discovery of differential and integral calculus in 17th–18th centuries**

- I. Newton (1642–1727)
- D. Bernoulli (1700–1782)
- L. Euler (1707–1783)

⋮

## Equations of motion of an incompressible fluid

We assume, for simplicity, that the density of the fluid is constant and equal to one.

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} &= \mathbf{f} + \operatorname{div} \mathbb{T} && \text{(conservation of momentunm)}, && (1.1) \\ \operatorname{div} \mathbf{u} &= 0 && \text{(conservation of mass)}, && (1.2) \end{aligned}$$

where $\mathbf{u}$ is the velocity, $\mathbf{f}$ is an external body force and $\mathbb{T}$ is the stress tensor.

**Constitutive equations:** provide the dependence of $\mathbb{T}$ on other quantities.

They can be deduced from **Stokes' postulates:**

a) the stress tensor $\mathbb{T}$ depends on velocity and its derivatives only through the rate of deformation tensor $\mathbb{D} := (\nabla \mathbf{u})_{\mathrm{sym}}$,

b) the stress tensor $\mathbb{T}$ does not explicitly depend on position $\mathbf{x}$ and time $t$,

c) the continuum is isotropic, i.e. it contains no preferred directions,

d) if the fluid is at rest then $\mathbb{T}$ is a multiple of the identity tensor $\mathbb{I}$ by a scalar.

One can show that c) follows from a more general postulate

c') the way tensor $\mathbb{T}$ depends on tensor $\mathbb{D}$ is frame indifferent.

Furthermore, using the postulates a), b), c') and d), one can derive that

$$\mathbb{T} \; = \; \alpha\mathbb{I} + \beta\mathbb{D} + \gamma\mathbb{D}^2,$$

where $\alpha$, $\beta$ and $\gamma$ may depend only on the *principle invariants of* $\mathbb{D}$ and the *state quantities,* which are the *pressure, the density and the temperature.*

In *Newtonian fluid,* $\mathbb{T}$ is supposed to depend *linearly* on $\mathbb{D}$.

From this, one can deduce that

$$\mathbb{T} \; = \; -p\mathbb{I} + 2\nu\mathbb{D},$$

where $\nu$ is the so called kinematic coefficient of viscosity.

We further suppose that $\nu = const. > 0$.

Substituting this form of $\mathbb{T}$ to the momentum equation (1.1), we obtain

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} \; = \; -\nabla p + \nu \Delta \mathbf{u}, \tag{1.1}$$

This equation, together with

$$\operatorname{div} \mathbf{u} \; = \; 0, \tag{1.2}$$

form the system of *Navier–Stokes equation.* Unknowns: $\mathbf{u}$ (velocity), $p$ (pressure).

The equations (1.1), (1.2) are usually studied in a spatial domain – let us denote it by $\Omega$ – in $\mathbb{R}^3$ and in some time interval – let it be $(0, T)$.

*Initial condition:*

$$\mathbf{u}(\,.\,, 0) = \mathbf{u}_0 \qquad \text{in } \Omega. \tag{1.3}$$

*Boundary conditions:* of various types, the most commonly used condition is

$$\mathbf{u} \;=\; \mathbf{0} \qquad \text{on } \partial\Omega \times (0, T). \tag{1.4}$$

Beginnings of the qualitative theory: **C. W. Oseen (1879–1944), J. Leray (1906–1998).**

J. Leray proved the global in time existence of *weak solutions* in $\Omega = \mathbb{R}^3$ in the 30–ties of the 20the century, similar results in other types of domain $\Omega$ appeared later.

The global in time existence of a strong solution is known only in some special cases, like in a 2D flow, in the case of "sufficiently small" $\mathbf{f}$ and $\mathbf{u}_0$, etc. The local in time existence of a strong solution is known e.g. for $\mathbf{u}_0 \in \mathbf{L}^3(\Omega)$, $\operatorname{div} \mathbf{u}_0 = 0$.

It is remarkable that although $p$ does not explicitly appear in the weak formulation of the problem (1.1)–(1.4), it is implicitly "hidden" in the formulation and plays an important role in the theory of the equations (1.1), (1.2).

## 2. Notation and some auxiliary results

| | | |
|---|---|---|
| $\Omega$ | ... | a domain in $\mathbb{R}^3$ |
| $\mathbf{n}$ | ... | the outer normal vector to $\partial\Omega$ |
| $\mathbf{C}^{\infty}_{0,\sigma}(\Omega)$ | ... | the linear space of infinitely differentiable divergence-free vector functions in $\Omega$, with a compact support in $\Omega$ |
| $\mathbf{L}^q_{\sigma}(\Omega)$ | ... | (for $1 < q < \infty$) is the closure of $\mathbf{C}^{\infty}_{0,\sigma}(\Omega)$ in $\mathbf{L}^q(\Omega)$ |
| $\mathbf{W}^{1,q}_{0,\sigma}(\Omega)$ | ... | the closure of $\mathbf{C}^{\infty}_{0,\sigma}(\Omega)$ in $\mathbf{W}^{1,q}(\Omega)$ |
| $\\| \, . \, \\|_q$ | ... | the norm in $L^q(\Omega)$ and in $\mathbf{L}^q(\Omega)$ |
| $\\| \, . \, \\|_{k,q}$ | ... | the norm in $W^{k,q}(\Omega)$ and in $\mathbf{W}^{k,q}(\Omega)$ (for $k \in \mathbb{N}$) |
| $\\| \, . \, \\|_{q;\,\Omega'}$ | ... | the norm in $W^{k,q}(\Omega)$ if $\Omega'$ differs from $\Omega$ |
| $( \, . \, , \, . \, )_2$ | ... | the scalar product in $L^2(\Omega)$ and in $\mathbf{L}^2(\Omega)$ |
| $( \, . \, , \, . \, )_{1,2}$ | ... | the scalar product in $W^{1,2}(\Omega)$ and in $\mathbf{W}^{1,2}(\Omega)$ |
| $q'$ | ... | the conjugate exponent to $q$ |

| | | |
|---|---|---|
| $\mathbf{W}_0^{-1,q'}(\Omega)$ | $\dots$ | the dual space to $\mathbf{W}_0^{1,q}(\Omega)$ |
| $\mathbf{W}_{0,\sigma}^{-1,q'}(\Omega)$ | $\dots$ | the dual space to $\mathbf{W}_{0,\sigma}^{1,q}(\Omega)$ |
| $\|\,.\,\|_{-1,q'}$ | $\dots$ | the norm in $\mathbf{W}_0^{-1,q'}(\Omega)$ |
| $\|\,.\,\|_{-1,q';\,\sigma}$ | $\dots$ | the norm in $\mathbf{W}_{0,\sigma}^{-1,q'}(\Omega)$ |
| $\langle\,.\,,\,.\,\rangle_\Omega$ | $\dots$ | the duality between elements of $\mathbf{W}_0^{-1,q'}(\Omega)$ and $\mathbf{W}_0^{1,q}(\Omega)$ |
| $\langle\,.\,,\,.\,\rangle_{\Omega,\sigma}$ | $\dots$ | the duality between elements of $\mathbf{W}_{0,\sigma}^{-1,q'}(\Omega)$ and $\mathbf{W}_{0,\sigma}^{1,q}(\Omega)$ |
| $\mathbf{W}_{0,\sigma}^{1,q}(\Omega)^\perp$ | $\dots$ | the space of annihilators of $\mathbf{W}_{0,\sigma}^{1,q}(\Omega)$ in $\mathbf{W}_0^{-1,q'}(\Omega)$ $=$ the space $\big\{\mathbf{f}\in\mathbf{W}_0^{-1,q'}(\Omega);\,\forall\boldsymbol{\varphi}\in\mathbf{W}_{0,\sigma}^{1,q}(\Omega):\langle\mathbf{f},\boldsymbol{\varphi}\rangle_\Omega=0\big\}$ |

**Remark.** Note that generally

$$\mathbf{W}_{0,\sigma}^{1,q}(\Omega)\ \subset\ \{\mathbf{v}\in\mathbf{W}_0^{1,q}(\Omega);\ \mathrm{div}\,\mathbf{v}=0\text{ a.e. in }\Omega\}.$$

The equality holds e.g. if $\Omega$ has a bounded Lipschitz boundary or $\Omega$ is a half-space, see [3] (the book by G. P. Galdi), Sec. III.4, for more details.

**Remark.** The Lebesgue space $\mathbf{L}^{q'}(\Omega)$ can be identified with a subspace of $\mathbf{W}_0^{-1,q'}(\Omega)$ so that if $\mathbf{f} \in \mathbf{L}^{q'}(\Omega)$ then

$$\langle \mathbf{f}, \boldsymbol{\varphi} \rangle_\Omega \;:=\; \int_\Omega \mathbf{f} \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} \tag{2.1}$$

for all $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,q}(\Omega)$. Similarly, $\mathbf{L}_\sigma^{q'}(\Omega)$ can be identified with a subspace of $\mathbf{W}_{0,\sigma}^{-1,q'}(\Omega)$ so that if $\mathbf{f} \in \mathbf{L}_\sigma^{q'}(\Omega)$ then

$$\langle \mathbf{f}, \boldsymbol{\varphi} \rangle_{\Omega,\sigma} \;:=\; \int_\Omega \mathbf{f} \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} \tag{2.2}$$

for all $\boldsymbol{\varphi} \in \mathbf{W}_{0,\sigma}^{1,q}(\Omega)$. If $\mathbf{f} \in \mathbf{L}_\sigma^{q'}(\Omega)$ and $\boldsymbol{\varphi} \in \mathbf{W}_{0,\sigma}^{1,q}(\Omega)$ then the dualities $\langle \mathbf{f}, \boldsymbol{\varphi} \rangle_\Omega$ and $\langle \mathbf{f}, \boldsymbol{\varphi} \rangle_{\Omega,\sigma}$ coincide, because they are expressed by the same integral.

**Remark.** If $\mathbf{f} \in \mathbf{L}^{q'}(\Omega)$ then the integral on the right hand side of (2.1) also defines a bounded linear functional on $\mathbf{W}_{0,\sigma}^{1,q}(\Omega)$.

This, however, does not mean that $\mathbf{L}^{q'}(\Omega)$ can be identified with a subspace of $\mathbf{W}_{0,\sigma}^{-1,q'}(\Omega)$.

The reason is, for instance, that the spaces $\mathbf{L}^{q'}(\Omega)$ and $\mathbf{W}_{0,\sigma}^{-1,q'}(\Omega)$ do not have the same zero element. (If $\psi$ is a non-constant function in $C_0^\infty(\Omega)$ then $\nabla\psi$ is a non-zero element of $\mathbf{L}^{q'}(\Omega)$, but it induces the zero element of $\mathbf{W}_{0,\sigma}^{-1,q'}(\Omega)$.)

**Operator $\mathscr{P}_{q'}$.** $\mathbf{W}^{1,q}_{0,\sigma}(\Omega)$ is a closed subspace of $\mathbf{W}^{1,q}_{0}(\Omega)$. If $\mathbf{f} \in \mathbf{W}^{-1,q'}_{0}(\Omega)$ then we denote by $\mathscr{P}_{q'}\mathbf{f}$ the restriction of $\mathbf{f}$ to $\mathbf{W}^{1,q}_{0,\sigma}(\Omega)$. Thus, $\mathscr{P}_{q'}\mathbf{f}$ is an element of $\mathbf{W}^{-1,q'}_{0,\sigma}(\Omega)$, defined by the equation

$$\langle \mathscr{P}_{q'}\mathbf{f}, \boldsymbol{\varphi} \rangle_{\Omega,\sigma} := \langle \mathbf{f}, \boldsymbol{\varphi} \rangle_{\Omega} \qquad \text{for all } \boldsymbol{\varphi} \in \mathbf{W}^{1,q}_{0,\sigma}(\Omega).$$

$\mathscr{P}_{q'}$ is a linear operator from $\mathbf{W}^{-1,q'}_{0}(\Omega)$ to $\mathbf{W}^{-1,q'}_{0,\sigma}(\Omega)$, $D(\mathscr{P}_{q'}) = \mathbf{W}^{-1,q'}_{0}(\Omega)$.

**Lemma 1.** *$\mathscr{P}_{q'}$ is a bounded operator from $\mathbf{W}^{-1,q'}_{0}(\Omega)$ to $\mathbf{W}^{-1,q'}_{0,\sigma}(\Omega)$. Its domain is the whole space $\mathbf{W}^{-1,q'}_{0}(\Omega)$, its range is the whole space $\mathbf{W}^{-1,q'}_{0,\sigma}(\Omega)$ and $\mathscr{P}_{q'}$ is not 1–1.*

**Proof.** Boundedness of $\mathscr{P}_{q'}$: Let $\mathbf{f} \in \mathbf{W}^{-1,q'}_{0}(\Omega)$. Then

$$\begin{aligned} \|\mathscr{P}_{q'}\mathbf{f}\|_{-1,q';\sigma} &= \sup_{\boldsymbol{\varphi}\in\mathbf{W}^{1,q}_{0,\sigma}(\Omega);\ \boldsymbol{\varphi}\neq\mathbf{0}} \frac{|\langle \mathscr{P}_{q'}\mathbf{f}, \boldsymbol{\varphi}\rangle_{\Omega,\sigma}|}{\|\boldsymbol{\varphi}\|_{1,q}} = \sup_{\boldsymbol{\varphi}\in\mathbf{W}^{1,q}_{0,\sigma}(\Omega);\ \boldsymbol{\varphi}\neq\mathbf{0}} \frac{|\langle \mathbf{f}, \boldsymbol{\varphi}\rangle_{\Omega}|}{\|\boldsymbol{\varphi}\|_{1,q}} \\ &\leq \sup_{\boldsymbol{\varphi}\in\mathbf{W}^{1,q}_{0}(\Omega);\ \boldsymbol{\varphi}\neq\mathbf{0}} \frac{|\langle \mathbf{f}, \boldsymbol{\varphi}\rangle_{\Omega}|}{\|\boldsymbol{\varphi}\|_{1,q}} = \|\mathbf{f}\|_{-1,q'}. \end{aligned}$$

Range of $\mathscr{P}_{q'}$: Let $\mathbf{g} \in \mathbf{W}_{0,\sigma}^{-1,q'}(\Omega)$. There exists (by the Hahn-Banach theorem) an extension of $\mathbf{g}$ from $\mathbf{W}_{0,\sigma}^{1,q}(\Omega)$ to $\mathbf{W}_{0}^{1,q}(\Omega)$, which we denote by $\mathbf{g}_{\mathrm{ext}}$. The extension is an element of $\mathbf{W}_{0}^{-1,q'}(\Omega)$, satisfying $\|\mathbf{g}_{\mathrm{ext}}\|_{-1,q'} = \|\mathbf{g}\|_{-1,q';\sigma}$ and

$$\langle \mathbf{g}_{\mathrm{ext}}, \boldsymbol{\varphi} \rangle_{\Omega} = \langle \mathbf{g}, \boldsymbol{\varphi} \rangle_{\Omega,\sigma} \qquad \text{for all } \boldsymbol{\varphi} \in \mathbf{W}_{0,\sigma}^{1,q}(\Omega). \tag{2.3}$$

This shows that $\mathbf{g} = \mathscr{P}_{q'}\mathbf{g}_{\mathrm{ext}}$. Consequently, $\mathbf{W}_{0,\sigma}^{-1,q'}(\Omega) = R(\mathscr{P}_{q'})$.

$\mathscr{P}_{q'}$ is not 1–1: Taking $\mathbf{f} = \nabla g$ for $g \in C_0^{\infty}(\Omega)$, we get

$$\langle \mathscr{P}_{q'}\mathbf{f}, \boldsymbol{\varphi} \rangle_{\Omega,\sigma} = \langle \mathbf{f}, \boldsymbol{\varphi} \rangle_{\Omega} = \int_{\Omega} \nabla q \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} = 0 \qquad \text{for all } \boldsymbol{\varphi} \in \mathbf{W}_{0,\sigma}^{1,q}(\Omega).$$

Hence $\mathscr{P}_{q'}$ is not 1–1. □

The next lemma tells us more on the space $\mathbf{W}^{1,q}_{0,\sigma}(\Omega)^{\perp}$ in the case when $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^3$. It comes from [7; Lemma II.2.2.2]. (The book by H. Sohr.)

**Lemma 2.** *Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^3$, $\Omega_0$ be a nonempty sub-domain of $\Omega$, $1 < q < \infty$ and $\mathbf{f}$ be a bounded linear functional on $\mathbf{W}^{1,q}_0(\Omega)$ that vanishes on $\mathbf{W}^{1,q}_{0,\sigma}(\Omega)$ (which means that $\mathbf{f} \in \mathbf{W}^{1,q}_{0,\sigma}(\Omega)^{\perp}$). Then there exists a unique function $\varphi \in L^{q'}(\Omega)$ such that $\int_{\Omega_0} \varphi \, \mathrm{d}\mathbf{x} = 0$,*

$$\langle \mathbf{f}, \boldsymbol{\psi} \rangle_{\Omega} = \int_{\Omega} \varphi \, \mathrm{div}\, \boldsymbol{\psi} \, \mathrm{d}\mathbf{x} \tag{2.4}$$

*for all $\boldsymbol{\psi} \in \mathbf{W}^{1,q}_0(\Omega)$*

$$\|\varphi\|_{q'} \leq c\, \|\mathbf{f}\|_{-1,q'} \tag{2.5}$$

*where $c = c(q, \Omega_0, \Omega)$.*

Formula (2.4) shows that $\mathbf{f} = \nabla\varphi$, where operator $\nabla$ acts on $\varphi$ in the sense of distributions. Thus, we may symbolically write $\mathbf{W}^{1,q}_{0,\sigma}(\Omega)^{\perp} = \nabla(L^{q'}(\Omega))$.

In order to characterize $\mathbf{W}_{0,\sigma}^{1,q}(\Omega)^\perp$ in the case of an arbitrary domain $\Omega$ in $\mathbb{R}^3$, we denote by $L^{q'}_{pot}(\Omega)$ the set of all $\varphi \in L^{q'}_{loc}(\Omega)$ such that $\nabla\varphi \in \mathbf{W}_0^{-1,q'}(\Omega)$.

**Lemma 3.** *If $\Omega$ is an arbitrary domain in $\mathbb{R}^3$, $\mathbf{f} \in \mathbf{W}_{0,\sigma}^{1,q}(\Omega)^\perp$ and $\Omega_0 \subset\subset \Omega$ is a nonempty sub-domain of $\Omega$ then there is a unique $\varphi \in L^{q'}_{pot}(\Omega)$ such that $\mathbf{f} = \nabla\varphi$ (the distributional gradient of $\varphi$) and $\int_{\Omega_0} \varphi \, \mathrm{d}\mathbf{x} = 0$.*

(Here and further on, $\Omega_0 \subset\subset \Omega$ means that $\Omega_0$ is a bounded sub-domain of $\Omega$ such that $\overline{\Omega_0} \subset \Omega$.) The lemma shows that $\mathbf{W}_{0,\sigma}^{1,q}(\Omega)^\perp = \nabla(L^{q'}_{pot}(\Omega))$.

The proof can be found in [5; Chap. 4] (the book “Fluids under Pressure”).

**The Helmholtz projection $P_{q'}$ and its relation to operator $\mathscr{P}_{q'}$.** Put $\mathbf{G}_{q'}(\Omega) := \{\nabla\psi \in \mathbf{L}^{q'}(\Omega);\ \psi \in W^{1,q'}_{loc}(\Omega)\}$. $\mathbf{G}_{q'}(\Omega)$ is a closed subspace of $\mathbf{L}^{q'}(\Omega)$, see [3; Exercise III.1.2] (the book by G. P. Galdi).

If each function $\mathbf{g} \in \mathbf{L}^{q'}(\Omega)$ can be uniquely expressed in the form

$$\mathbf{g} \ = \ \mathbf{v} + \nabla\psi$$

for some $\mathbf{v} \in \mathbf{L}^{q'}_{\sigma}(\Omega)$ and $\nabla\psi \in \mathbf{G}_{q'}(\Omega)$, which is equivalent to the validity of the decomposition

$$\mathbf{L}^{q'}(\Omega) \ = \ \mathbf{L}^{q'}_{\sigma}(\Omega) \oplus \mathbf{G}_{q'}(\Omega), \tag{2.6}$$

then we write

$$\mathbf{v} \ = \ P_{q'}\mathbf{g}.$$

Decomposition (2.6) is called the *Helmholtz decomposition* and the operator $P_{q'}$ is called the *Helmholtz projection.*

If $q' = 2$ then the Helhholtz decomposition exists on an arbitrary domain $\Omega$ and $P_2$ is the orthogonal projection of $\mathbf{L}^2(\Omega)$ onto $\mathbf{L}^2_{\sigma}(\Omega)$.

If $q' \neq 2$ then the Helmholtz decomposition exists e.g. if $\Omega$ is a domain of the class $C^2$ (see [3; Section III.1]).

Assume, for a while, that the Helmholtz decomposition of $\mathbf{L}^{q'}(\Omega)$ exists.

What is the relation between the operators $\mathscr{P}_{q'}$ and $P_{q'}$? Let $\mathbf{g} \in \mathbf{L}^{q'}(\Omega)$.

Recall that $\mathscr{P}_{q'} : \mathbf{W}_0^{-1,q'}(\Omega) \to \mathbf{W}_{0,\sigma}^{-1,q'}(\Omega)$, while $P_{q'} : \mathbf{L}^{q'}(\Omega) \to \mathbf{L}_\sigma q'$.

Let $\mathbf{g} \in \mathbf{L}^{q'}(\Omega)$. (Hence $\mathbf{g}$ can also be treated as an element of $\mathbf{W}_0^{-1,q'}(\Omega)$.) One can show that

$$\langle \mathscr{P}_{q'}\mathbf{g}, \boldsymbol{\varphi} \rangle_{\Omega,\sigma} \;=\; \langle P_{q'}\mathbf{g}, \boldsymbol{\varphi} \rangle_{\Omega,\sigma}$$

for all $\boldsymbol{\varphi} \in \mathbf{W}_{0,\sigma}^{1,q}(\Omega)$.

From this, we observe that the Helmholtz projection $P_{q'}$ coincides with the restriction of $\mathscr{P}_{q'}$ to $\mathbf{L}^{q'}(\Omega)$.

## 3. A weak solution of the Navier–Stokes IBVP – three equiv. definitions

**Classical form of the Navier–Stokes IBVP:** For $T > 0$, we consider

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p &= \nu \Delta \mathbf{u} + \mathbf{f} && \text{in } Q_T := \Omega \times (0,T), && (3.1)\\ \operatorname{div} \mathbf{u} &= 0 && \text{in } Q_T, && (3.2)\\ \text{boundary condition:} \qquad \mathbf{u} &= \mathbf{0} && \text{on } \Gamma_T := \partial\Omega \times (0,T) && (3.3)\\ \text{initial condition:} \qquad \mathbf{u} &= \mathbf{u}_0 && \text{in } \Omega \times \{0\}. && (3.4) \end{aligned}$$

**Definition 1 of a weak solution of the Navier-Stokes IBVP (3.1)–(3.4).** *Given* $\mathbf{u}_0 \in \mathbf{L}^2_\sigma(\Omega)$ *and* $\mathbf{f} \in L^2(0,T;\, \mathbf{W}_0^{-1,2}(\Omega))$. *A function* $\mathbf{u} \in L^\infty(0,T;\, \mathbf{L}^2_\sigma(\Omega)) \cap L^2(0,T;\, \mathbf{W}^{1,2}_{0,\sigma}(\Omega))$ *is said to be a weak solution to the problem (3.1)–(3.4) if*

$$\int_0^T \int_\Omega \big[-\mathbf{u} \cdot \partial_t \boldsymbol{\phi} + \nu \nabla \mathbf{u} : \nabla \boldsymbol{\phi} + \mathbf{u} \cdot \nabla \mathbf{u} \cdot \boldsymbol{\phi}\big] \, \mathrm{d}\mathbf{x} \, \mathrm{d}t = \int_\Omega \mathbf{u}_0 \cdot \boldsymbol{\phi}(\mathbf{x}, 0) \, \mathrm{d}\mathbf{x} + \int_0^T \langle \mathbf{f}, \boldsymbol{\phi} \rangle_\Omega \, \mathrm{d}t \qquad (3.5)$$

*for all* $\boldsymbol{\phi} \in C^\infty\big([0,T];\ \mathbf{W}^{1,2}_{0,\sigma}(\Omega)\big)$ *such that* $\boldsymbol{\phi}(T) = \mathbf{0}$.

Define $\mathcal{A} : \mathbf{W}_0^{1,2}(\Omega) \to \mathbf{W}_0^{-1,2}(\Omega)$ and $\mathcal{B} : \left[\mathbf{W}_0^{1,2}(\Omega)\right]^2 \to \mathbf{W}_0^{-1,2}(\Omega)$ by the equations

$$\begin{aligned}
\big\langle \mathcal{A}\mathbf{v}, \boldsymbol{\varphi} \big\rangle_\Omega \; &:= \; \int_\Omega \nabla\mathbf{v} : \nabla\boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} && \text{for } \mathbf{v}, \boldsymbol{\varphi} \in \mathbf{W}_0^{1,2}(\Omega), \\
\big\langle \mathcal{B}(\mathbf{v}, \mathbf{w}), \boldsymbol{\varphi} \big\rangle_\Omega \; &:= \; \int_\Omega \mathbf{v} \cdot \nabla\mathbf{w} \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} && \text{for } \mathbf{v}, \mathbf{w}, \boldsymbol{\varphi} \in \mathbf{W}_0^{1,2}(\Omega).
\end{aligned}$$

Obviously, operator $\mathcal{A}$ is one-to-one and

$$\|\mathcal{A}\mathbf{v}\|_{-1,2} \; \leq \; \|\nabla\mathbf{v}\|_2. \tag{3.6}$$

The bilinear operator $\mathcal{B}$ satisfies

$$\begin{aligned}
\|\mathcal{B}(\mathbf{v}, \mathbf{w})\|_{-1,2} \; &= \sup_{\boldsymbol{\varphi} \in \mathbf{W}_0^{1,2}(\Omega),\ \boldsymbol{\varphi} \neq \mathbf{0}} \frac{|\langle \mathcal{B}(\mathbf{v}, \mathbf{w}), \boldsymbol{\varphi} \rangle_\Omega|}{\|\boldsymbol{\varphi}\|_{1,2}} \; = \sup_{\boldsymbol{\varphi} \in \mathbf{W}_0^{1,2}(\Omega),\ \boldsymbol{\varphi} \neq \mathbf{0}} \frac{|(\mathbf{v} \cdot \nabla\mathbf{w},\ \boldsymbol{\varphi})_2|}{\|\boldsymbol{\varphi}\|_{1,2}} \\
&\leq \sup_{\boldsymbol{\varphi} \in \mathbf{W}_0^{1,2}(\Omega),\ \boldsymbol{\varphi} \neq \mathbf{0}} \frac{\|\mathbf{v}\|_2^{1/2} \, \|\mathbf{v}\|_6^{1/2} \, \|\nabla\mathbf{w}\|_2 \, \|\boldsymbol{\varphi}\|_6}{\|\boldsymbol{\varphi}\|_{1,2}} \; \leq \; c \, \|\mathbf{v}\|_2^{1/2} \, \|\nabla\mathbf{v}\|_2^{1/2} \, \|\nabla\mathbf{w}\|_2.
\end{aligned} \tag{3.7}$$

Let $\mathbf{u}$ be a weak solution of the IBVP (3.1)–(3.4) in the sense of Definition 1. It follows from the estimates (3.6) and (3.7) that

$$\mathcal{A}\mathbf{u} \in L^2(0,T;\, \mathbf{W}_0^{-1,2}(\Omega)) \quad \text{and} \quad \mathcal{B}(\mathbf{u},\mathbf{u}) \in L^{4/3}(0,T;\, \mathbf{W}_0^{-1,2}(\Omega)). \tag{3.8}$$

Considering function $\boldsymbol{\phi}$ in (3.5) in the form $\boldsymbol{\phi}(\mathbf{x},t) = \boldsymbol{\varphi}(\mathbf{x})\,\vartheta(t)$ where $\boldsymbol{\varphi} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ and $\vartheta \in C_0^\infty((0,T))$, we deduce that $\mathbf{u}$ satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(\mathbf{u},\boldsymbol{\varphi})_2 + \nu\,\big\langle \mathcal{A}\mathbf{u},\boldsymbol{\varphi}\big\rangle_\Omega + \big\langle \mathcal{B}(\mathbf{u},\mathbf{u}),\boldsymbol{\varphi}\big\rangle_\Omega \;=\; \langle \mathbf{f},\boldsymbol{\varphi}\rangle_\Omega \tag{3.9}$$

a.e. in $(0,T)$, where the derivative of $(\mathbf{u},\boldsymbol{\varphi})_2$ means the derivative in the sense of distributions.

It follows from (3.8) that $\langle \mathcal{A}\mathbf{u},\boldsymbol{\varphi}\rangle_\Omega \in L^2(0,T)$ and $\langle \mathcal{B}(\mathbf{u},\mathbf{u}),\boldsymbol{\varphi}\rangle_\Omega \in L^{4/3}(0,T)$. Since $\langle \mathbf{f},\boldsymbol{\varphi}\rangle_\Omega \in L^2(0,T)$, we observe from (3.9) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(\mathbf{u},\boldsymbol{\varphi})_2 \text{ (in the sense of distributions) } \in\; L^{4/3}(0,T).$$

Hence $(\mathbf{u},\boldsymbol{\varphi})_2$ is (after a possible redefinition on a set of measure zero) a continuous function in $[0,T)$. Now, one can deduce from (3.5) that

$$(\mathbf{u},\boldsymbol{\varphi})_2\big|_{t=0} = \;(\mathbf{u}_0,\boldsymbol{\varphi})_2 \qquad \text{for all } \boldsymbol{\varphi} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega). \tag{3.10}$$

**Definition 2 of a weak solution of the Navier-Stokes IBVP (3.1)–(3.4).** *Given* $\mathbf{u}_0 \in \mathbf{L}^2_\sigma(\Omega)$ *and* $\mathbf{f} \in L^2(0,T;\ \mathbf{W}_0^{-1,2}(\Omega))$. *Find* $\mathbf{u} \in L^\infty(0,T;\ \mathbf{L}^2_\sigma(\Omega)) \cap L^2(0,T;\, \mathbf{W}^{1,2}_{0,\sigma}(\Omega))$ *(called the weak solution) such that, for each* $\boldsymbol{\varphi} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$, $\mathbf{u}$ *satisfies the equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(\mathbf{u},\boldsymbol{\varphi})_2 + \nu\,\big\langle \mathcal{A}\mathbf{u},\boldsymbol{\varphi}\big\rangle_\Omega + \big\langle \mathcal{B}(\mathbf{u},\mathbf{u}),\boldsymbol{\varphi}\big\rangle_\Omega \;=\; \langle \mathbf{f},\boldsymbol{\varphi}\rangle_\Omega \tag{3.9}$$

*a.e. in* $(0,T)$ *and the initial condition*

$$(\mathbf{u},\boldsymbol{\varphi})_2\big|_{t=0} = \;(\mathbf{u}_0,\boldsymbol{\varphi})_2 \qquad \textit{for all } \boldsymbol{\varphi} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega). \tag{3.10}$$

**Lemma 4.** *Let* $\mathbf{X}$ *be a Banach space with the dual* $\mathbf{X}^*$, $\langle\, .\,,\, .\,\rangle$ *be the duality between* $\mathbf{X}^*$ *and* $\mathbf{X}$, $-\infty < a < b < \infty$ *and* $\mathbf{u}$, $\mathbf{g} \in L^1(a,b;\, \mathbf{X})$. *Then the following three conditions are equivalent:*

*1)* $\mathbf{u}$ *is a.e. in* $(a,b)$ *equal to a primitive function of* $\mathbf{g}$*, which means that*

$$\mathbf{u}(t) = \boldsymbol{\xi} + \int_a^t \mathbf{g}(s)\, \mathrm{d}s \quad \textit{for some } \boldsymbol{\xi} \in \mathbf{X} \textit{ and a.a. } t \in (a,b),$$

*2)* $$\int_a^b \vartheta'(t)\, \mathbf{u}(t)\, \mathrm{d}t = -\int_a^b \vartheta(t)\, \mathbf{g}(t)\, \mathrm{d}t \quad \textit{for all } \vartheta \in C_0^\infty((a,b)),$$

*3)* $$\frac{\mathrm{d}}{\mathrm{d}t}\, \langle \boldsymbol{\eta}, \mathbf{u} \rangle = \langle \boldsymbol{\eta}, \mathbf{g} \rangle \quad \textit{in the sense of distributions in } (a,b) \textit{ for each } \boldsymbol{\eta} \in \mathbf{X}^*.$$

*If the conditions 1) – 3) are fulfilled then* $\mathbf{u}$ *is a.e. in* $(a,b)$ *equal to a continuous function from* $[a,b]$ *to* $\mathbf{X}$.

See Lemma III.1.1 in [8] (the book by R. Temam).

Note that if functions $\mathbf{u}$ and $\mathbf{g}$ are related as in item 2) then $\mathbf{g}$ is called the *distributional derivative* of $\mathbf{u}$ with respect to $t$ and it is usually denoted by $\mathbf{u}'$.

Equation (3.9) can also be written in the equivalent form

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(\mathbf{u},\boldsymbol{\varphi})_2 + \nu\,\big\langle \mathscr{P}_2\mathcal{A}\mathbf{u},\boldsymbol{\varphi}\big\rangle_{\Omega,\sigma} + \big\langle \mathscr{P}_2\mathcal{B}(\mathbf{u},\mathbf{u}),\boldsymbol{\varphi}\big\rangle_{\Omega,\sigma} \;=\; \big\langle \mathscr{P}_2\mathbf{f},\boldsymbol{\varphi}\big\rangle_{\Omega,\sigma}. \tag{3.11}$$

Let us denote by $(\mathbf{u}')_\sigma$ the distributional derivative with respect to $t$ of $\mathbf{u}$, as a function from $(0,T)$ to $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$. Applying Lemma $4$ (with $\mathbf{X} = \mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$ and $\mathbf{X}^* = \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$), we deduce that equation (3.11) is equivalent to

$$(\mathbf{u}')_\sigma + \nu\,\mathscr{P}_2\mathcal{A}\mathbf{u} + \mathscr{P}_2\mathcal{B}(\mathbf{u},\mathbf{u}) \;=\; \mathscr{P}_2\mathbf{f}, \tag{3.12}$$

which is an equation in $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$, satisfied a.e. in the time interval $(0,T)$.
It shows that $(\mathbf{u}')_\sigma \in L^{4/3}(0,T;\ \mathbf{W}^{-1,2}_{0,\sigma}(\Omega))$.
Hence $\mathbf{u}$ coincides a.e. in $(0,T)$ with a continuous function from $[0,T)$ to $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$.

**Definition 3 of a weak solution of the Navier-Stokes IBVP (3.1)–(3.4).** *Given* $\mathbf{u}_0 \in \mathbf{L}^2_\sigma(\Omega)$ *and* $\mathbf{f} \in L^2(0,T;\,\mathbf{W}^{-1,2}_0(\Omega))$. *Function* $\mathbf{u} \in L^\infty(0,T;\ \mathbf{L}^2_\sigma(\Omega)) \cap L^2(0,T;\,\mathbf{W}^{1,2}_{0,\sigma}(\Omega))$ *is called a weak solution to the IBVP (3.1)–(3.4) if* $\mathbf{u}$ *satisfies equation (3.12) a.e. in the interval* $(0,T)$ *and the initial condition (3.4), where* $\mathbf{u}|_{t=0}$ *is the value of the aforementioned continuous function at time* $t=0$.

**Remark.** We have shown that $\mathbf{u}$ coincides a.e. in $(0,T)$ with a continuous function from $[0,T)$ to $\mathbf{W}_{0,\sigma}^{-1,2}(\Omega)$.

This, however, does not imply that $\mathbf{u}$ coincides a.e. in $(0,T)$ with a continuous function from $[0,T)$ to $\mathbf{W}_0^{-1,2}(\Omega)$.

It is connected with the fact that the derivative $(\mathbf{u}')_\sigma$ in equation (3.12) is the distributional derivative with respect to $t$ of $\mathbf{u}$, as a function from $(0,T)$ to $\mathbf{W}_{0,\sigma}^{-1,2}(\Omega)$ and not the distributional derivative with respect to $t$ of $\mathbf{u}$, as a function from $(0,T)$ to $\mathbf{W}_0^{-1,2}(\Omega)$.

As it is important to distinguish between these two derivatives, we use the different notation: while the first derivative is denoted by $(\mathbf{u}')_\sigma$, the second is denoted just by $\mathbf{u}'$. We can formally write $(\mathbf{u}')_\sigma = \mathscr{P}_2\mathbf{u}'$.

# 4. An associated pressure – existence, structure, uniqueness

**Projections $E^{1,2}$ and $E^{-1,2}$.** Recall that $\mathbf{W}_0^{1,2}(\Omega)$ is a Hilbert space with the scalar product

$$(\mathbf{u},\mathbf{v})_{1,2} = \int_\Omega \big(\nabla\mathbf{u} : \nabla\mathbf{v} + \mathbf{u}\cdot\mathbf{v}\big)\,\mathrm{d}\mathbf{x} = \big\langle(\mathcal{A}+I)\mathbf{u},\mathbf{v}\big\rangle_\Omega.$$

By analogy, $\mathbf{W}_0^{-1,2}(\Omega)$ is a Hilbert space with the scalar product

$$(\mathbf{f},\mathbf{g})_{-1,2} = \big\langle\mathbf{f},(\mathcal{A}+I)^{-1}\mathbf{g}\big\rangle_\Omega = \big((\mathcal{A}+I)^{-1}\mathbf{f},(\mathcal{A}+I)^{-1}\mathbf{g}\big)_{1,2}. \tag{4.1}$$

Denote by $E^{1,2}$ the orthogonal projection in $\mathbf{W}_0^{1,2}(\Omega)$ such that

$$\ker E^{1,2} = \mathbf{W}_{0,\sigma}^{1,2}(\Omega), \tag{4.2}$$

and by $E^{-1,2}$ the adjoint projection in $\mathbf{W}_0^{-1,2}(\Omega)$.

It follows from (4.2) that the range of $E^{-1,2}$ is $\mathbf{W}_{0,\sigma}^{1,2}(\Omega)^\perp$.

Due to (4.1) and the orthogonality of $E^{1,2}$, we have, for $\mathbf{f}\in\mathbf{W}_0^{-1,2}(\Omega)$ and $\boldsymbol{\psi}\in\mathbf{W}_0^{1,2}(\Omega)$

$$\big\langle\mathbf{f},E^{1,2}\boldsymbol{\psi}\big\rangle_\Omega = \big((\mathcal{A}+I)^{-1}\mathbf{f},E^{1,2}\boldsymbol{\psi}\big)_{1,2} = \big(E^{1,2}(\mathcal{A}+I)^{-1}\mathbf{f},\boldsymbol{\psi}\big)_{1,2}.$$

However, using (4.1) and the fact that $E^{-1,2}$ is adjoint to $E^{1,2}$, we also have

$$\big\langle \mathbf{f}, E^{1,2}\boldsymbol{\psi}\big\rangle_{\Omega} \;=\; \big\langle E^{-1,2}\mathbf{f}, \boldsymbol{\psi}\big\rangle_{\Omega} \;=\; \big((\mathcal{A}+I)^{-1}E^{-1,2}\mathbf{f}, \boldsymbol{\psi}\big)_{1,2}$$

for all $\mathbf{f} \in \mathbf{W}_0^{-1,2}(\Omega)$ and $\boldsymbol{\psi} \in \mathbf{W}_0^{1,2}(\Omega)$.

This shows that

$$E^{1,2}(\mathcal{A}+I)^{-1} \;=\; (\mathcal{A}+I)^{-1}E^{-1,2}. \tag{4.3}$$

Applying this identity and the orthogonality of projection $E^{1,2}$, we get

$$\begin{aligned}
\big(E^{-1,2}\mathbf{f}, \mathbf{g}\big)_{-1,2} \;&=\; \big((\mathcal{A}+I)^{-1}E^{-1,2}\mathbf{f}, (\mathcal{A}+I)^{-1}\mathbf{g}\big)_{1,2} \\
&= \big(E^{1,2}(\mathcal{A}+I)^{-1}\mathbf{f}, (\mathcal{A}+I)^{-1}\mathbf{g}\big)_{1,2} \\
&=\; \big((\mathcal{A}+I)^{-1}\mathbf{f}, E^{1,2}(\mathcal{A}+I)^{-1}\mathbf{g}\big)_{1,2} \\
&=\; \big((\mathcal{A}+I)^{-1}\mathbf{f}, (\mathcal{A}+I)^{-1}E^{-1,2}\mathbf{g}\big)_{1,2} \\
&=\; \big(\mathbf{f}, E^{-1,2}\mathbf{g}\big)_{-1,2}
\end{aligned}$$

for all $\mathbf{f},\, \mathbf{g} \in \mathbf{W}_0^{-1,2}(\Omega)$, which shows that projection $E^{-1,2}$ is orthogonal, too.

Finally, let $\phi \in C_0^\infty(\Omega)$. Then

$$(\mathcal{A}+I)\nabla\phi \equiv \nabla(-\Delta+I)\phi \;\in\; \mathbf{W}_{0,\sigma}^{1,2}(\Omega)^\perp.$$

Hence

$$E^{-1,2}(\mathcal{A}+I)\nabla\phi \;=\; (\mathcal{A}+I)\nabla\phi.$$

Moreover, applying (4.3), we also have

$$E^{-1,2}(\mathcal{A}+I)\nabla\phi \;=\; (\mathcal{A}+I)E^{1,2}\phi.$$

As $\mathcal{A}+I$ is a one-to-one operator from $\mathbf{W}_0^{1,2}(\Omega)$ to $\mathbf{W}_0^{-1,2}(\Omega)$, the last two equalities show that

$$E^{1,2}\nabla\phi \;=\; \nabla\phi \qquad \text{for all } \phi \in C_0^\infty(\Omega). \tag{4.4}$$

We have $\mathbf{f} \in L^2(0,T;\mathbf{W}_0^{-1,2}(\Omega))$ in the definition of a weak solution.
We may identify $\mathbf{f}$ with a distribution in $Q_T$, acting on functions $\boldsymbol{\phi} \in \mathbf{C}_0^\infty(Q_T)$ through the formula

$$\big\langle\!\big\langle \mathbf{f}, \boldsymbol{\phi} \big\rangle\!\big\rangle_{Q_T} \;:=\; \int_0^T \big\langle \mathbf{f}(t), \boldsymbol{\phi}(\,.\,,t) \big\rangle_\Omega \, \mathrm{d}t. \tag{4.5}$$

($\langle\!\langle\,.\,,\,.\,\rangle\!\rangle_{Q_T}$ denotes the action of a distribution in $Q_T$ on a function from $C_0^\infty(Q_T)$ or $\mathbf{C}_0^\infty(Q_T)$.)

**Definition of an associated pressure.** *Let* $\mathbf{u}$ *be a weak solution to the Navier–Stokes IBVP (3.1)–(3.4). If there exists a distribution* $p$ *in* $Q_T$ *such that* $\mathbf{u}$ *and* $p$ *satisfy the Navier-Stokes equation*

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p \;=\; \nu\Delta\mathbf{u} + \mathbf{f} \tag{3.1}$$

*in the sense of distributions in* $Q_T$ *then* $p$ *is called a pressure, associated with the weak solution* $\mathbf{u}$.

**Existence of an associated pressure.** Let $\mathbf{u}$ be a weak solution to the IBVP (3.1)–(3.4). Due to Lemma 4 , item 1), equation (3.12) is equivalent to

$$\mathbf{u}(t) - \mathbf{u}(0) + \int_0^t \mathscr{P}_2\big[\nu\mathcal{A}\mathbf{u} + \mathcal{B}(\mathbf{u},\mathbf{u}) - \mathbf{f}\big]\,\mathrm{d}\tau \;=\; \mathbf{0}$$

for a.a. $t \in (0,T)$. Since $\mathbf{u}(t)$ and $\mathbf{u}(0)$ are in $\mathbf{L}^2_\sigma(\Omega)$, they coincide with $\mathscr{P}_2\mathbf{u}(t)$ and $\mathscr{P}_2\mathbf{u}(0)$, respectively. Hence

$$\mathscr{P}_2\bigg(\mathbf{u}(t) - \mathbf{u}(0) + \int_0^t \big[\nu\mathcal{A}\mathbf{u} + \mathcal{B}(\mathbf{u},\mathbf{u}) - \mathbf{f}\big]\,\mathrm{d}\tau\bigg) \;=\; \mathbf{0}.$$

Define $\mathbf{F}(t) \in \mathbf{W}_0^{-1,2}(\Omega)$ by the formula

$$\mathbf{F}(t) \;:=\; \mathbf{u}(t) - \mathbf{u}(0) + \int_0^t \big[\nu\mathcal{A}\mathbf{u} + \mathcal{B}(\mathbf{u},\mathbf{u}) - \mathbf{f}\big]\,\mathrm{d}\tau. \tag{4.6}$$

$\mathbf{F}(t)$ is an element of $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)^\perp$. Hence $E^{-1,2}\mathbf{F}(t) = \mathbf{F}(t)$ and $(I - E^{-1,2})\mathbf{F}(t) = \mathbf{0}$. Thus,

$$\big\langle \mathbf{F}(t), (I - E^{1,2})\boldsymbol{\psi}\big\rangle_\Omega \;=\; \big\langle (I - E^{-1,2})\mathbf{F}(t), \boldsymbol{\psi}\big\rangle_\Omega \;=\; 0$$

for all $\boldsymbol{\psi} \in \mathbf{W}^{1,2}_0(\Omega)$.

It means that

$$(I - E^{-1,2})\mathbf{u}(t) - (I - E^{-1,2})\mathbf{u}(0) + \int_0^t (I - E^{-1,2})\left[\nu\mathcal{A}\mathbf{u} + \mathcal{B}(\mathbf{u},\mathbf{u}) - \mathbf{f}\right] \mathrm{d}\tau \;=\; \mathbf{0}$$

holds as an equation in $\mathbf{W}_0^{-1,2}(\Omega)$. Applying Lemma 4 (with $\mathbf{X} = \mathbf{W}_0^{-1,2}(\Omega)$), we get

$$\left[(I - E^{-1,2})\mathbf{u}\right]' + (I - E^{-1,2})\left[\nu\mathcal{A}\mathbf{u} + \mathcal{B}(\mathbf{u},\mathbf{u}) - \mathbf{f}\right] \;=\; \mathbf{0},$$

where $[(I - E^{-1,2})\mathbf{u}]'$ is the distributional derivative with respect to $t$ of $(I - E^{-1,2})\mathbf{u}$, as a function from $(0,T)$ to $\mathbf{W}_0^{-1,2}(\Omega)$. This yields

$$\mathbf{u}' + \nu\mathcal{A}\mathbf{u} + \mathcal{B}(\mathbf{u},\mathbf{u}) \;=\; \mathbf{f} + [E^{-1,2}\mathbf{u}]' + \nu E^{-1,2}\mathcal{A}\mathbf{u} + E^{-1,2}\mathcal{B}(\mathbf{u},\mathbf{u}) - E^{-1,2}\mathbf{f}. \quad (4.7)$$

Let $\Omega_0 \subset\subset \Omega$ be a non-empty domain. By Lemma 3, there exist unique $p_0(t)$, $p_1(t)$, $p_2(t)$, $p_3(t)$ in $L^2_{pot}(\Omega)$ such that

$$\begin{aligned} \nabla p_0(t) &= -E^{-1,2}\mathbf{u}(t), & \nabla p_1(t) &= -\nu E^{-1,2}\mathcal{A}\mathbf{u}(t), \\ \nabla p_2(t) &= -E^{-1,2}\mathcal{B}\big(\mathbf{u}(t),\mathbf{u}(t)\big), & \nabla p_3(t) &= E^{-1,2}\mathbf{f}(t) \end{aligned} \quad (4.8)$$

and $\int_{\Omega_0} p_i(t)\, \mathrm{d}\mathbf{x} = 0$ ($i = 0,1,2,3$) for a.a. $t \in (0,T)$.

Using (3.8) and the boundedness of projection $E^{-1,2}$, we get

$$
\begin{aligned}
&\nabla p_0 \in L^\infty(0,T;\ \mathbf{W}_0^{-1,2}(\Omega)), \qquad && \nabla p_1 \in L^2(0,T;\ \mathbf{W}_0^{-1,2}(\Omega)),\\
&\nabla p_2 \in L^{4/3}(0,T;\ \mathbf{W}_0^{-1,2}(\Omega)), \qquad && \nabla p_3 \in L^2(0,T;\ \mathbf{W}_0^{-1,2}(\Omega)).
\end{aligned}
\tag{4.9}
$$

Hence

$$
\begin{aligned}
&p_0 \in L^\infty(0,T;\ L^2_{loc}(\Omega)), \qquad && p_1 \in L^2(0,T;\ L^2_{loc}(\Omega)),\\
&p_2 \in L^{4/3}(0,T;\ L^2_{loc}(\Omega)), \qquad && p_3 \in L^2(0,T;\ L^2_{loc}(\Omega)).
\end{aligned}
\tag{4.10}
$$

Equation (4.7) shows that if we put

$$
p \ := \ \partial_t p_0 + p_1 + p_2 + p_3, \tag{4.11}
$$

where $\partial_t p_0$ is the distributional derivative of $p_0$ with respect to $t$ then

$$
\begin{aligned}
&\int_0^T \int_\Omega \big[-\mathbf{u}\cdot\boldsymbol{\psi}\,\eta'(t) + \nu\nabla\mathbf{u} : \nabla\boldsymbol{\psi}\,\eta(t) + \mathbf{u}\cdot\nabla\mathbf{u}\cdot\boldsymbol{\psi}\,\eta(t)\big]\ \mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\qquad = \int_0^T \langle \mathbf{f}, \boldsymbol{\psi}\rangle_\Omega\,\eta(t)\ \mathrm{d}t + \int_0^T\int_\Omega p\,\mathrm{div}\,\boldsymbol{\psi}\,\eta(t)\ \mathrm{d}\mathbf{x}\,\mathrm{d}t
\end{aligned}
$$

for all functions $\boldsymbol{\psi} \in \mathbf{W}_0^{1,2}(\Omega)$ and $\eta \in C_0^\infty((0,T))$.

Since the set of all finite linear combinations of functions of the type $\boldsymbol{\psi}(\mathbf{x})\,\eta(t)$, where $\boldsymbol{\psi} \in \mathbf{W}_0^{1,2}(\Omega)$ and $\eta \in C_0^\infty((0,T))$, is dense in $\mathbf{C}_0^\infty(Q_T)$ in the topology of $L^{4/3}(0,T;\mathbf{W}_0^{-1,2}(\Omega))$, we also have

$$\int_0^T \int_\Omega \big[-\mathbf{u}\cdot\partial_t\boldsymbol{\varphi}+\nu\nabla\mathbf{u} : \nabla\boldsymbol{\varphi}+\mathbf{u}\cdot\nabla\mathbf{u}\cdot\boldsymbol{\varphi}\big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \;=\; \int_0^T \langle\mathbf{f},\boldsymbol{\varphi}\rangle_\Omega\,\mathrm{d}t+\int_0^T\int_\Omega p\,\mathrm{div}\,\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$

for all $\boldsymbol{\varphi} \in \mathbf{C}_0^\infty(Q_T)$. This shows that the pair $\mathbf{u}$, $p$ satisfies the Navier-Stokes equation (3.1) in the sense of distributions in $Q_T$.

For a.a. $t \in (0,T)$, the functions $p_0(t)$ and $p_1(t)$ are harmonic in $\Omega$. This follows from the identities

$$\begin{aligned}\int_\Omega p_0(t)\,\Delta\phi\,\mathrm{d}\mathbf{x} \;&=\; -\big\langle\nabla p_0(t),\nabla\phi\big\rangle_\Omega \;=\; \big\langle E^{-1,2}\mathbf{u}(t),\nabla\phi\big\rangle_\Omega \;=\; \big\langle\mathbf{u}(t),E^{1,2}\nabla\phi\big\rangle_\Omega\\ &=\; \big\langle\mathbf{u}(t),\nabla\phi\big\rangle_\Omega \;=\; \int_\Omega\mathbf{u}(t)\cdot\nabla\phi\,\mathrm{d}\mathbf{x} \;=\; 0 \qquad \text{(for all } \phi\in C_0^\infty(\Omega)\text{)}.\end{aligned}$$

(We have used (4.4).) Hence, by Weyl's lemma, $p_0(t)$ is a harmonic function in $\Omega$. The fact that $p_1(t)$ is harmonic can be proven similarly.

**Uniqueness of the associated pressure up to an additive distribution of the form (4.12).** If $G$ is a distribution in $(0,T)$ and $\psi \in \mathbf{C}_0^\infty(Q_T)$ then we define a distribution $g$ in $Q_T$ by the formula

$$\big\langle\!\big\langle g, \psi \big\rangle\!\big\rangle_{Q_T} \;:=\; \Big\langle G,\, \int_\Omega \psi \; \mathrm{d}\mathbf{x} \Big\rangle_{(0,T)}, \tag{4.12}$$

where $\langle G,\, .\, \rangle_{(0,T)}$ denotes the action of distribution $G$ on a function from $C_0^\infty((0,T))$. Obviously, if $\boldsymbol{\phi} \in \mathbf{C}_0^\infty(Q_T)$ then

$$\big\langle\!\big\langle \nabla g, \boldsymbol{\phi} \big\rangle\!\big\rangle_{Q_T} \;=\; -\big\langle\!\big\langle g, \operatorname{div} \boldsymbol{\phi} \big\rangle\!\big\rangle_{Q_T} \;=\; -\Big\langle G,\, \int_\Omega \operatorname{div} \boldsymbol{\phi} \; \mathrm{d}\mathbf{x} \Big\rangle_{(0,T)} \;=\; 0, \tag{4.13}$$

because $\int_\Omega \operatorname{div} \boldsymbol{\phi}(\,.\,, t)\; \mathrm{d}\mathbf{x} = 0$ for all $t \in (0,T)$. Thus, $p + g$ (where $p$ is given by (4.11)) is a pressure, associated with the weak solution $\mathbf{u}$ to the IBVP (3.1)–(3.4), too.

On the other hand, if $p + g$ is a pressure, associated with the weak solution $\mathbf{u}$, then $g$ satisfies

$$0 \;=\; \big\langle\!\big\langle \nabla g, \boldsymbol{\phi} \big\rangle\!\big\rangle_{Q_T} \;=\; -\big\langle\!\big\langle g, \operatorname{div} \boldsymbol{\phi} \big\rangle\!\big\rangle_{Q_T} \qquad \text{for all } \boldsymbol{\phi} \in \mathbf{C}_0^\infty(Q_T). \tag{4.14}$$

For $h \in C_0^\infty((0,T))$, define

$$\big\langle G, h \big\rangle_{(0,T)} \;:=\; \big\langle\!\big\langle g, \psi \big\rangle\!\big\rangle_{Q_T}, \tag{4.15}$$

where $\psi \in C_0^\infty(Q_T)$ is chosen so that $h(t) = \int_\Omega \psi(\mathbf{x},t)\,\mathrm{d}\mathbf{x}$ for all $t \in (0,T)$.

The definition of the distribution $G$ is independent of the concrete choice of function $\psi$ due to these reasons:

Let $\psi_1$ and $\psi_2$ be two functions from $C_0^\infty(Q_T)$ such that

$$h(t) \;=\; \int_\Omega \psi_1(\mathbf{x},t)\,\mathrm{d}\mathbf{x} \;=\; \int_\Omega \psi_2(\mathbf{x},t)\,\mathrm{d}\mathbf{x} \qquad \text{for } t \in (0,T).$$

Denote by $G_1$, respectively $G_2$, the distribution, defined by formula (4.15) with $\psi = \psi_1$, respectively $\psi = \psi_2$.

Since $\operatorname{supp}(\psi_1 - \psi_2)$ is a compact subset of $Q_T$ and

$$\int_\Omega [\psi_1(\,.\,,t) - \psi_2(\,.\,,t)]\,\mathrm{d}\mathbf{x} \;=\; 0 \qquad \text{for all } t \in (0,T),$$

there exists a function $\boldsymbol{\phi} \in \mathbf{C}_0^\infty(Q_T)$ such that $\operatorname{div}\boldsymbol{\phi} = \psi_1 - \psi_2$ in $Q_T$.

Then

$$\big\langle G_1 - G_2, h \big\rangle_{(0,T)} \; := \; \big\langle\!\big\langle g, \psi_1 - \psi_2 \big\rangle\!\big\rangle_{Q_T} \; = \; \big\langle\!\big\langle g, \operatorname{div} \boldsymbol{\phi} \big\rangle\!\big\rangle_{Q_T},$$

which is equal to zero due to (4.14). Formula (4.15) and the identity $h(t) = \int_\Omega \psi(\mathbf{x}, t)\, \mathrm{d}\mathbf{x}$ show that the distribution $g$ has the form (4.12).

The next theorem summarizes the derived results:

**Theorem 1.** *Let* $\mathbf{u}$ *be a weak solution to the Navier-Stokes IBVP (3.1)–(3.4). Then there exists an associated pressure* $p$ *(as a distribution in* $Q_T$*) of the form (4.11), where* $p_0$*,* $p_2$*,* $p_3$*,* $p_4$ *satisfy (4.8)–(4.10). Moreover,*

*1) if* $\Omega_0 \subset\subset \Omega$ *then the functions* $p_0(t), \dots, p_3(t)$ *can be chosen uniquely so that they satisfy the additional conditions* $\int_{\Omega_0} p_i(t)\ \mathrm{d}\mathbf{x} = 0$ *for* $i = 0, 1, 2, 3$ *and a.a.* $t \in (0, T)$*,*

*2) the functions* $p_0(t)$ *and* $p_1(t)$ *are harmonic in* $\Omega$ *for a.a.* $t \in (0, T)$*,*

*3)* $p + g$ *is also a pressure, associated with the weak solution* $\mathbf{u}$*, if and only if* $g$ *is a distribution of the form (4.12).*

**Remark.** If $\Omega$ is a bounded Lipschitz domain then the statement of Theorem 1 can be improved so that $L^2_{loc}(\Omega)$ is replaced by $L^2(\Omega)$ in (4.10) and the choice $\Omega_0 = \Omega$ is also permitted in statement 2). This is enabled by Lemma 2, which shows that the range of projection $E^{-1,2}$ coincides with $\nabla(L^2(\Omega))$.

Analogous results for the Navier–Stokes equations with Navier's boundary conditions

$$\text{a)}\ \ \mathbf{u}\cdot\mathbf{n} = 0, \qquad \text{b)}\ \ [\mathbb{T}\cdot\mathbf{n}]_\tau + \gamma\mathbf{u} = \mathbf{0},$$

see [6] (Š. Nečasová, J. Neustupa, P. Kučera), 2020.

**Example.** We give an example of a simple distributional solution to the system (3.1), (3.2), that is not smooth in dependence on $t$ and the associated pressure cannot be identified with a function from $L^1_{loc}(Q_T)$.

Although the solution does not satisfy the boundary condition (3.3), the example sheds light on the reasons why the pressure generally exists only as a distribution and not as a function.

Let $\psi \in \mathbf{W}^{2,2}(\Omega)$ be a harmonic function in $\Omega$ and $a(t) \in L^\infty(0,T)$. Put

$$\begin{aligned} \mathbf{u}(\mathbf{x},t) &:= a(t)\,\nabla\psi(\mathbf{x}), \\ p(\mathbf{x},t) &:= -a'(t)\,\psi(\mathbf{x}) - a^2(t)\,\frac{|\psi(\mathbf{x})|^2}{2}. \end{aligned}$$

for $\mathbf{x} \in \Omega$ and $0 \leq t < T$.

Function $\mathbf{u}$ is divergence–free and the pair $\mathbf{u}$, $p$ satisfies the Navier-Stokes equation (3.1) (with $\mathbf{f} = \mathbf{0}$) in the sense of distributions in $Q_T$.

If $a(t)$ is chosen so that the derivative $a'(t)$ exists only as a distribution in $(0,T)$ that cannot be identified with a function from $L^1_{loc}((0,T))$ then $p$ is a distribution in $Q_T$ that cannot be identified with a function from $L^1_{loc}(Q_T)$.

# 5. An associated pressure in the case of a smooth domain

The next theorem follows from Theorem 3.1 in [4] (Y. Giga and H. Sohr, 1991):

**Theorem 2.** *LET $\Omega$ be a bounded or exterior domain in $\mathbb{R}^3$ with the boundary of the class $C^{2+(h)}$ for some $h > 0$, or a half-space in $\mathbb{R}^3$ or the whole space $\mathbb{R}^3$.*

*LET* $0 < T \leq \infty,\ 1 < s < \frac{3}{2},\ 1 < r < 2,\ 2/r + 3/s = 4,\ \mathbf{f} \in L^r(0,T;\, \mathbf{L}^s(\Omega)) \cap L^2(0,T;\, \mathbf{L}^2(\Omega))$ *and* $\mathbf{u}_0 \in \mathbf{W}^{2,s}(\Omega) \cap \mathbf{W}^{1,s}_{0,\sigma}(\Omega) \cap \mathbf{L}^2_\sigma(\Omega)$.

*LET $\mathbf{u}$ be a weak solution to the Navier-Stokes IBVP (3.1)–(3.4) and $p$ be an associated pressure.*

*THEN $\mathbf{u} \in L^r(0,T_0;\ \mathbf{W}^{2,s}(\Omega))$ for each $0 < T_0 \leq T$, $T_0 < \infty$ and function $p$ can be chosen so that it belongs to $L^r(0,T;\ L^{3s/(3-s)}(\Omega))$. Functions $\mathbf{u}$, $p$ satisfy the equations (3.1), (3.2) a.e. in $Q_T$.*

**Remark.** The pressure $p$ is determined uniquely up to an additive function $g \in L^r(0,T)$.

The choice $r = \frac{5}{3}$, $s = \frac{15}{14}$ in Theorem 2 yields $p \in L^{5/3}(Q_T)$.

## 6. A pressure, associated with a suitable or dissipative weak solution

**Briefly on suitable weak solutions.** If $\Omega$ is a smooth domain then a series of authors have shown that the problem (3.1)–(3.4) has the so called *suitable weak solution,* which is a pair of functions $\mathbf{u}$, $p$ such that $\mathbf{u}$ is a weak solution, $p$ is an associated pressure, and $\mathbf{u}$, $p$ satisfy the *generalized energy inequality*

$$2\nu \int_0^T \int_\Omega |\nabla \mathbf{u}|^2 \, \phi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \ \le \ \int_0^T \int_\Omega \big[ \, |\mathbf{u}|^2 \, \big(\partial_t \phi + \nu \Delta \phi\big) + \big(|\mathbf{u}|^2 + 2p\big) \, \mathbf{u} \cdot \nabla \phi \big] \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \\ + 2 \int_0^T \big\langle \mathbf{f}, \mathbf{u}\phi \big\rangle_\Omega \, \mathrm{d}t \tag{6.1}$$

for every non-negative scalar function $\phi$ with a compact support in $Q_T$. This inequality is also often called the *local (*or *localized) energy inequality.*

In order to give a reasonable sense to the integral of $2p \, \mathbf{u} \cdot \nabla \phi$ in (6.1), it is necessary to include some assumptions on the integrability of $p$ to the definition of a suitable weak solution: most of the authors consider $p \in L^{3/2}(Q_T)$ or $p \in L^{5/3}(Q_T)$.

The first results of this type: L. Caffarelli, R. Kohn, L. Nirenberg [1], 1982.

The local energy inequality enables one to derive a series of "local regularity criteria", i.e. criteria for regularity of the suitable weak solution at just one point.

Recall that $(\mathbf{x}_0, t_0)$ is said to be a *regular point* of solution $\mathbf{u}$ if there exists a neighborhood $U(\mathbf{x}_0, t_0)$ in $Q_T$ such that $\mathbf{u}$ is essentially bounded in $U(\mathbf{x}_0, t_0)$.

**Example:** *There exists $\epsilon > 0$ such that if $(\mathbf{x}_0, t_0) \in Q_T$ and $\mathbf{u}$ is a suitable weak solution to the problem (3.1)–(3.4), satisfying*

$$\limsup_{r \to 0+} \frac{1}{r} \int_{t_0 - r^2}^{t_0} \int_{B_r(\mathbf{x}_0)} |\nabla \mathbf{u}|^2 \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \; < \; \epsilon \tag{6.2}$$

*then $(\mathbf{x}_0, t_0)$ is a regular point of solution $\mathbf{u}$.*

There exist many modifications or generalizations of this criterion. For example, the same proposition also holds if $|\nabla \mathbf{u}|$ is replaced by $|\mathbf{curl}\ \mathbf{u} \times (\mathbf{u}/|\mathbf{u}|)|$, see J. Wolf [9].

The criterion (6.2) can be used in order to prove that the set of hypothetic *singular points* of solution $\mathbf{u}$ (i.e. points of $Q_T$ that are not regular) has the one–dimensional Hausdorff measure of the set of singular points is also zero.

**Remark.** Note that if $\mathbf{f}$ is e.g. in $\mathbf{L}^{10/7}(Q_T)$ then $\mathbf{f}\cdot\mathbf{u} \in L^1(Q_T)$. Thus, the quantity

$$\mu \;:=\; -\partial_t|\mathbf{u}|^2 + \nu\Delta|\mathbf{u}|^2 - 2\nu|\nabla\mathbf{u}|^2 - \operatorname{div}\,(|\mathbf{u}|^2\mathbf{u}) - 2\operatorname{div}\,(p\mathbf{u}) + 2\mathbf{f}\cdot\mathbf{u} \tag{6.3}$$

is well defined as a distribution in $Q_T$. In this case, (6.1) means that $\mu \geq 0$ in $Q_T$, which means that $\langle\langle \mu, \phi\rangle\rangle_{Q_T} \geq 0$ for all $\phi \in C_0^\infty(Q_T)$, $\phi \geq 0$.

**Briefly on a dissipative weak solution.** This notion comes from D. Chamorro, P.-G. Lemarié-Riusset and K. Mayoufi [2], 2018.

The authors show that if $\mathbf{u}$, $p$ is a distributional solution of the Navier-Stokes system (3.1), (3.2) in $Q := B_\rho(\mathbf{x}_0) \times (a,b)$ (where $\rho > 0$ and $-\infty < a < b < \infty$) such that $\mathbf{u} \in L^\infty(a,b;\, \mathbf{L}^2(B_\rho(\mathbf{x}_0))) \cap L^2(a,b;\, \mathbf{W}^{1,2}(B_\rho(\mathbf{x}_0)))$ then the product $p\mathbf{u}$ exists as a distribution in $Q$.

Then they define a *dissipative weak solution* $\mathbf{u}$, $p$ to the system (3.1), (3.2) in $Q$ to be a distributional solution of (3.1), (3.2) in $Q$, such that $\mathbf{u} \in L^\infty(a,b;\, \mathbf{L}^2(B_\rho(\mathbf{x}_0))) \cap L^2(a,b;\, \mathbf{W}^{1,2}(B_\rho(\mathbf{x}_0)))$ and $\mu \geq 0$ in $Q$.

It is proven in [2] that if $\mathbf{u}$ is a dissipative weak solution in some neighborhood of $(\mathbf{x}_0, t_0)$ then it satisfies the same regularity criterion (6.2) as the suitable weak solution.

# References


1. L. Caffarelli, R. Kohn, L. Nirenberg: Partial regularity of suitable weak solutions of the Navier-Stokes equations. *Comm. on Pure and Appl. Math.* **35** (1982), 771–831.
2. D. Chamorro, P.-G. Lemarié-Riusset, K. Mayoufi: The role of pressure in the partial regularity theory for weak solutions of the Navier-Stokes equations. *Arch. Rat. Mech. Anal.* **228** (2018), 237–277.
3. G. P. Galdi: *An Introduction to the Mathematical Theory of the Navier-Stokes Equations.* 2nd edition, Springer 2011.
4. Y. Giga, H. Sohr: Abstract $L^p$-estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains. *J. Functional Analysis* **102** (1991), 72–94.
5. J. Neustupa: The role of pressure in the theory of weak solutions of the Navier–Stokes equations. In *Fluids Under Pressure,* eds. T. Bodnár, G. P. Galdi, Š. Nečasová, Birkhäuser, 2020, pp. 349–416.
6. J. Neustupa, Š. Nečasová, P. Kučera: A pressure associated with a weak solution to the Navier–Stokes equations with Navier's boundary conditions. *J. Math. Fluid Mech.* **22** (2020), article number 37, https://doi.org/10.1007/s00021-020-00500-y.
7. H. Sohr: *The Navier-Stokes Equations. An Elementary Functional Analytic Approach.* Birkhäuser Advanced Texts, Basel-Boston-Berlin 2001.
8. R. Temam: *Navier-Stokes Equations.* North-Holland, Amsterdam-New York-Oxford 1977.
9. J. Wolf: A direct proof of the Caffarelli-Kohn-Nirenberg theorem. *Parabolic and Navier-Stokes equations, Banach Center Publ.* **81**, Part 2, Polish Acad. of Sci., Inst. Math., Warsaw 2008, 533–552.