Scattering below first excited solitons for non-radial NLS with potential

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1. Introduction and Main result
   - Setup and Previous works
   - Main result

2. Outline of the proof
   - The existence of the first excited states
   - Global dynamics below first excited states

cubic NLS in 3D

NLS equation with linear potential

$$\begin{cases} i\partial_t u + Hu = |u|^2 u, & (t,x) \in \mathbb{R} \times \mathbb{R}^3, \\ u(0) = u_0 \in H^1(\mathbb{R}^3), \end{cases} \tag{NLS}$$

where

- $V(x) : \mathbb{R}^3 \to \mathbb{R}$ is a linear potential.
- $H = -\Delta + V$ has one simple negative eigenvalue $e_0 < 0$.

Goal

Global behavior of solutions with small mass and energy less than the first excited states.

The potential-less case: $V = 0$

Let us recall the results on the potential-less case $V = 0$:

$$\begin{cases} i\partial_t u - \Delta u = |u|^2 u, & (t,x) \in \mathbb{R} \times \mathbb{R}^3, \\ u(0) = u_0 \in H^1(\mathbb{R}^3). \end{cases} \qquad (\mathrm{NLS}_0)$$

Recent progress

- The study begins for solutions close to special solutions such as the zero and the ground state Q. Recently, more general solutions are treated with a help of variational argument.
- As a result, several sharp criterion are obtained in terms of the conserved quantities.

In the sequel, Q denotes the positive radial solution to

$$-\Delta Q + Q = Q^3.$$

Previous results for the case $V = 0$

Functional

$$\mathbb{M}(\varphi) := \int_{\mathbb{R}^3} \frac{1}{2}|\varphi(x)|^2 dx, \qquad \text{(mass)}$$

$$\mathbb{H}_0(\varphi) := \int_{\mathbb{R}^3} \frac{1}{2}|\nabla\varphi(x)|^2 dx,$$

$$\mathbb{G}(\varphi) := \int_{\mathbb{R}^3} \frac{1}{4}|\varphi(x)|^4 dx.$$

$$\mathbb{E}_0(\varphi) := \mathbb{H}_0(\varphi) - \mathbb{G}(\varphi). \qquad \text{(energy)}$$

$$\begin{aligned} \mathbb{K}_{0,2}(\varphi) &:= \partial_{\alpha=1}(\mathbb{E}_0(e^{\frac{3}{2}\alpha}\varphi(e^{\alpha}\cdot))) \\ &= 2\mathbb{H}_0(\varphi) - 3\mathbb{G}(\varphi). \end{aligned}$$

Previous results for the case $V = 0$

Holmer-Roudenko, Duyckaerts-Holmer-Roudenko, Akahori-Nawa

The set

$$B := \{\varphi \in H^1(\mathbb{R}^3) \mid \mathbb{M}(\varphi)\mathbb{E}_0(\varphi) < \mathbb{M}(Q)\mathbb{E}_0(Q)\} \subset H^1$$

splits into two disjoint subsets according to the sign of $\mathbb{K}_{0,2}$.

- If $\mathbb{K}_{0,2}(u_0) < 0$ then the solution $u(t)$ blows up for both time directions (in finite or infinite time).
- If $\mathbb{K}_{0,2}(u_0) \geqslant 0$ then the solution $u(t)$ is global and scatters for both time directions.

Remark [Duyckaerts-Roudenko, Nakanishi-Schlag]
Global dynamics in $\mathbb{M}(u)\mathbb{E}_0(u) < \mathbb{M}(Q)\mathbb{E}_0(Q) + \varepsilon$
(cf. 9-set theorem).

Scattering and Space-time boundedness

We say a solution $u(t)$ to (NLS_0) scatters forward in time if $\exists u_+ \in H^1$ s.t.

$$u(t) \to e^{-it\Delta} u_+ \quad \text{in } H^1$$

as $t \to \infty$.

Equivalent characterization (cf. Kato '94)

A solution $u(t)$ to (NLS_0) scatters forward in time iff

$$\|u\|_{L_t^8([0,T_{\max}),L_x^4(\mathbb{R}^3))} < \infty$$

(global existence $T_{\max} = \infty$ also follows).
Linear solutions satisfy this bound (cf. Strichartz est.)

Motivation

stability of ground state Q

In most cases, the global dynamics for large data is studied for equation with unstable ground state. However, in view of the physical model, it is natural to have a stable ground state.

- As for the standard NLS

$$i\partial_t u - \Delta u = |u|^{p-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d,$$

  the ground state Q is stable if and only if $p < 1 + 4/d$. The equation in this range is called mass-subcritical.
- However, the analysis of global dynamics for mass-subcritical equations is hard due to the fact that the scaling critical space has negative regularity. (e.g. smallness in H^1 implies nothing on the global dynamics).

previous attempts

Global dynamics on mass-subcritical ($\mathrm{NLS}_{\mathbf{0}}$)

- Weighted spaces
  M. '14, M. '15, Killip-M.-Murphy-Visan '17;
- Sobolev space with negative regularity and radial symmetry
  Killip-M.-Murphy-Visan '19;
- Fourier Lubesgue and Bourgain-Morrey spaces
  Segata-M. '18, M. '16

Today's model

Stable ground states and a linear potential

The situation is also created by adding a linear potential.
Due to the presence of a linear potential which yields a negative eigenvalue of H, (NLS) has stable ground states and unstable first excited states (at least under small mass constraint)

(NLS$_0$) ($V = 0$)		(NLS) ($V \neq 0$)
$\mathbf{0}$	$\rightarrow$	stable ground states
Q	$\rightarrow$	unstable first excited states

cubic NLS in 3D

Let us consider our model:

$$\begin{cases} i\partial_t u + Hu = |u|^2 u, & (t,x) \in \mathbb{R} \times \mathbb{R}^3, \\ u(0) = u_0 \in H^1(\mathbb{R}^3), \end{cases} \tag{NLS}$$

Related works

- Gustafson-Nakanishi-Tsai '04,
  Scattering to a ground state for small (in H^1) solutions
- Nakanishi '17, '17
  Global dynamics of solutions with small mass and energy less than that for the first excited states $+\varepsilon$, under radial symmetry.
- Many other results without a negative eigenvalue.

Assumption on the potential

For simplicity, we assume the following:

Assumption

V is a Schwartz function such that

(A1) $H = -\Delta + V$ *has one negative simple eigenvalue* $e_0 < 0$. *There is no other eigenvalues.* 0 *is not a resonance of* H;

(A2) $V(0) = \inf_{x \in \mathbb{R}^3} V(x) < 0$.

Remark

- Let $\psi \in \mathcal{S}(\mathbb{R}^3)$ be a positive radially decreasing nonzero function. Then, $a\psi$ satisfies the condition for a negative constant a in a suitable range,
- (A2) is essentially the choice of the coordinate.

Functional

We introduce functionals involving the linear potential V.

Functional

$$\mathbb{H}_V(\varphi) := \int_{\mathbb{R}^3} \left(\frac{1}{2} |\nabla \varphi(x)|^2 + \frac{1}{2} V(x) |\varphi(x)|^2 \right) dx,$$

また,

$$\mathbb{E}_V(\varphi) := \mathbb{H}_V(\varphi) - \mathbb{G}(\varphi) \qquad \text{(energy)}$$

$$\begin{aligned} \mathbb{K}_{V,2}(\varphi) &:= \partial_{\alpha=1}(\mathbb{E}_V(e^{\frac{3}{2}\alpha}\varphi(e^{\alpha}\cdot))) \\ &= 2\mathbb{H}_0(\varphi) - 3\mathbb{G}(\varphi) - \int_{\mathbb{R}^3} \frac{1}{2} x \cdot \nabla V(x) |\varphi(x)|^2 dx. \end{aligned}$$

The notation is consistent: They coincide with those with subscription "0" when $V = 0$.

Ground state energy $\mathscr{E}_0(\mu)$ and First excited energy $\mathscr{E}_1(\mu)$

A set of solitions

$$\mathscr{S} := \{\varphi \in \boldsymbol{H}^1(\mathbb{R}^3) \mid \exists\omega \in \mathbb{R} \text{ s.t. } (\boldsymbol{H}+\omega)\varphi = |\varphi|^2\varphi\}.$$

For any $\varphi \in \mathscr{S}$ and the corresponding number ω, the function $e^{-i\omega t}\varphi$ is an exact solution to (NLS) (**soliton**).

$\mathscr{E}_0(\mu)$ and $\mathscr{E}_1(\mu)$

For a prescribed value of mass $\mu > 0$, we let

$$\mathscr{E}_0(\mu) := \inf\{\mathbb{E}_V(\phi) \mid \phi \in \mathscr{S},\, \mathbb{M}(\phi) = \mu\},$$
$$\mathscr{E}_1(\mu) := \inf\{\mathbb{E}_V(\phi) \mid \phi \in \mathscr{S},\, \mathbb{M}(\phi) = \mu,\, \mathbb{E}_V(\phi) > \mathscr{E}_0(\mu)\}.$$

The ground state

(e_0, ϕ_0): the e.v and the normalized e.f. of H $(\mathbb{M}(\phi_0) = 1)$

Ground state $\Phi[z]$ (Gustafson-Nakanishi-Tsai)

One has

$$\mathscr{E}_0(\mu) = e_0\mu + O(\mu^2) \qquad (\mu \downarrow 0).$$

Further, $\exists \mu_* > 0$ s.t. if $0 < \mu < \mu_*$ then $\exists \Phi[z] \in \mathscr{S}$ s.t.

$$\mathbb{E}_V(\Phi[z]) = \mathscr{E}_0(\mu),$$

where $z \in \mathbb{C}$ is a complex-valued parameter. Further, we have

$$\Phi[e^{i\theta}z] = e^{i\theta}\Phi[z], \qquad (\Phi[z], \phi_0)_{L^2} = 2z$$

and

$$\Phi[z] = z\phi_0 + o(|z|^2)$$

Main result1: Existence of the first excited states

Theorem (M.-Murphy-Segata)

$\exists \mu_* > 0$ s.t.if $\mu < \mu_*$ then $\mathscr{E}_1(\mu) < \infty$ and $\exists \phi_1 \in \mathscr{S}$ s.t. $\mathbb{E}_V(\phi_1) = \mathscr{E}_1(\mu)$. Further,

$$\mu^{-1} \lesssim \mathscr{E}_1(\mu) \leqslant \mu^{-1}\mathbb{M}(Q)\mathbb{E}_0(Q) + (V(0) + o(1))\mu. \quad (**)$$

as $\mu \downarrow 0$.

Remark

- For $\mu > 0$ small, one has

$$\mu\mathscr{E}_1(\mu) < \mathbb{M}(Q)\mathbb{E}_0(Q).$$

This implies that the first excited state energy is less than the energy of the ground state for (NLS_0) (with the same mass).

Main result 2: global dynamics below first excited states

Theorem (M.-Murphy-Segata)

$\exists \mu_{**} > 0$ s.t. the set

$$\mathscr{B} := \{u_0 \in H^1(\mathbb{R}^3) \mid \mathbb{M}(u_0) \leqslant \mu_{**},\ \mathbb{E}_V(u_0) < \mathscr{E}_1(\mathbb{M}(u_0))\}$$

splits into two disjoint subsets according to the validity of

$$\|\nabla u_0\|_{L^2} \geqslant 1 \quad \text{and} \quad \mathbb{K}_{V,2}(u_0) < 0. \tag{BC}$$

Further,

- *If $u_0 \in \mathscr{B}$ and (BC) is true then the sol. $u(t)$ blows up for both time direction (in finite or infinite time).*
- *If $u_0 \in \mathscr{B}$ and (BC) is false then the sol. $u(t)$ is global and scatters to a ground state for both time directions, i.e., $\exists z(t)$ s.t. $\|u(t) - \Phi[z(t)]\|_{L^8(\mathbb{R};L^4)} < \infty$.*

Outline of the proof of the first theorem

The proof is divided into three steps.

1. Introduce $\tilde{\mathscr{E}}_{\mathbf{1}}(\boldsymbol{\mu})$, another characterization of $\mathscr{E}_{\mathbf{1}}(\boldsymbol{\mu})$;
2. Prove $\tilde{\mathscr{E}}_{\mathbf{1}}(\boldsymbol{\mu})$ obeys the estimate $(**)$;
3. Construct a minimizer to $\tilde{\mathscr{E}}_{\mathbf{1}}(\boldsymbol{\mu})$.

Step 1

Definition

$$\tilde{\mathscr{E}}_1(\mu) := \inf\left\{ \mathbb{I}_V(\varphi) \;\middle|\; \begin{array}{l} \varphi \in H^1,\ \mathbb{M}(\varphi) \leqslant \mu, \\ \mathbb{K}_{V,2}(\varphi) \leqslant 0,\ \mathbb{G}(\varphi) \geqslant 1 \end{array} \right\}$$

where

$$\begin{aligned} \mathbb{I}_V(\varphi) &:= \mathbb{E}_V(\varphi) - \tfrac{1}{2}\mathbb{K}_{V,2}(\varphi) \\ &= \tfrac{1}{2}\mathbb{G}(\varphi) + \tfrac{1}{4}\int (x \cdot \nabla V + 2V)|\varphi|^2 dx \end{aligned}$$

Remark

• It is easy to see that $\tilde{\mathscr{E}}_1(\mu) < \infty$ (i.e., the nonemptyness of the set where the infimum is considered);

• Minimization of $\mathbb{E}_V = \mathbb{H}_V - \mathbb{G}$ is hard since it is not coercive. $\mathbb{I}_V(\varphi)$ is much easier to handle.

Step 2

Lemma (Estimate $(**)$ for $\tilde{\mathscr{E}}_1(\mu)$)

$\forall \varepsilon > 0$, $\exists \mu_*(\varepsilon)$ *s.t.* $\forall \mu \in (0, \mu_*)$

$$\tilde{\mathscr{E}}_1(\mu) \leqslant \mu^{-1}\mathbb{M}(Q)\mathbb{E}_0(Q) + (V(0) + \varepsilon)\,\mu$$

(Idea of the proof) By comparison with the value of $\mathbb{I}_V$ for a specific function.
Substitute $\varphi = Q_\lambda := \lambda^{-1}Q(\cdot/\lambda)$ into

$$\mathbb{I}_V(\varphi) = \frac{1}{2}\mathbb{G}(\varphi) + \frac{1}{4}\int (x \cdot \nabla V + 2V)|\varphi|^2 dx.$$

Then,

$$\tilde{\mathscr{E}}_1(\mu) \leqslant \mathbb{I}_V(Q_\lambda) = \mu^{-1}\mathbb{M}(Q)\mathbb{E}_0(Q) + (V(0) + o(1))\,\mu$$

as $\lambda \downarrow 0$, where $\mu = \mathbb{M}(Q)\lambda = \mathbb{M}(Q_\lambda)$.

Step 3

Lemma

For $\mu > 0$ small, there exists a minimizer to $\tilde{\mathscr{E}}_1(\mu)$.

(Sketch of the proof) Take a minimizing sequence $\{v_n\}$.
$(\mathbb{E}_V(v_n) \to \tilde{\mathscr{E}}_1(\mu)$, $\mathbb{M}(v_n) \to \mu$, $\mathbb{K}_{V,2}(v_n) \to 0$ $)$
We apply a profile decomposition of H^1 bounded sequence based on the Lieb-type compactness theorem for $H^1 \hookrightarrow L^4$:
$\exists \psi_j \in H^1$, $\exists y_n^j \in \mathbb{R}^3$ s.t. upto a subseq., $\forall J \geqslant 1$

$$v_n = \psi_0 + \sum_{j=1}^{J} \psi_j(\cdot - y_n^j) + R_n^J.$$

Further, $\lim_{n\to\infty} |y_n^j| = \infty$,

$$|y_n^{j_1} - y_n^{j_2}| \to \infty \ (j_1 \neq j_2), \qquad \lim_{J\to\infty} \overline{\lim_{n\to\infty}} \left\| R_n^J \right\|_{L^4} = 0.$$

Moreover, we have the decoupling (in)equality:

$$\sum_{j=0}^{\infty} \mathbb{M}(\psi_j) \leqslant \mu, \quad \mathbb{K}_{V,2}(\psi_0) + \sum_{j=1}^{\infty} \mathbb{K}_{0,2}(\psi_j) \leqslant 0,$$

$$\tilde{\mathscr{E}}_1(\mu) = \mathbb{I}_V(\psi_0) + \sum_{j=1}^{\infty} \mathbb{I}_0(\psi_j).$$

The effect of V is negligible for the profiles shifted to the spacial infinity.

Three cases

- $\psi_j = 0$ $(\forall j \geqslant 1) \Rightarrow$ conclusion (compactness)!;
- $\psi_j \neq 0$ for one $j \geqslant 1 \Rightarrow$ precluded by $(**)$;
- $\psi_j \neq 0$ for multiple $j \geqslant 1 \Rightarrow$ precluded more easily.

Remark If we put the radial symmetry, the compactness $\psi_j = 0$ $(\forall j \geqslant 1)$ immediately follows from the radial Sobolev.

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Global existence

The variational characterization of $\mathscr{E}_1$ gives us the following.

Theorem

$\exists \mu_{**} > 0$ s.t. the set

$$\mathscr{B} := \{u_0 \in H^1(\mathbb{R}^3) \mid \mathbb{M}(u_0) \leqslant \mu_{**},\ \mathbb{E}_V(u_0) < \mathscr{E}_1(\mathbb{M}(u_0))\}$$

splits into two disjoint subsets according to the validity of

$$\|\nabla u_0\|_{L^2} \geqslant 1 \quad \text{and} \quad \mathbb{K}_{V,2}(u_0) < 0 \qquad \text{(BC)}$$

Further

- *If $u_0 \in \mathscr{B}$ and (BC) is true then the sol. $u(t)$ satisfies (BC) on its lifespan.*
- *If $u_0 \in \mathscr{B}$ and (BC) is flase then the sol. $u(t)$ is global and belongs to $L^\infty(\mathbb{R}, H^1)$.*

On scattering to a ground state

The blowup under the condition (BC) is standard (cf. [Akahori-Nawa]).

The main part of the proof is to establish scattering to ground state in the latter case.

Strategy

1. Define the curve $z(t)$, the parameter for the ground state part, from $u(t)$,
2. Write $u(t) = \Phi[z(t)] + \eta$ and apply Kenig-Merle type argument to the radiation part η.

Decomposition into a sum of a ground state and a radiation

Extraction of a ground-state part

$\exists \mu_{***} > 0$ s.t. any $u \in H^1$ with $M(u) < \mu_{***}$ is uniquely decomposed into

$$u = \Phi[z] + \eta, \qquad \eta \in P_c[z]H^1,$$

where

$$P_c[z]H^1 := \{f \in H^1 \mid \mathrm{Re}(if, \partial_{z_j}\Phi[z]) = 0 \ (j = 1, 2)\}.$$

Remark:
∂_{z_1}, ∂_{z_2} are the partial derivatives obtained by regarding $\Phi[z]$ as a function of $(z_1, z_2) \in \mathbb{R}^2$ via $z = z_1 + iz_2$.

Remark:
The scattering to a ground state is characterized as $\|\eta\|_{L_t^8(\mathbb{R}, L_x^4)} < \infty$.

a PDE-ODE system (1/2)

Let us derive a PDE-ODE system for the sosliton part z and the radiation part η (cf. Gustafson-Nakanishi-Tsai).

An inconvenience and a remedy

The radiation part η belongs to a time-dependent space $P_c[z]H^1$.

Letting $\xi := P_c[0]\eta$, we fix the space to $P_cH^1 := P_c[0]H^1$.

- $P_c[0]f = f - \frac{1}{\sqrt{2}}\phi_0(f, \frac{1}{\sqrt{2}}\phi_0)$.
- $P_c[0]|_{P_c[z]H^1}$ is invertible if $|z| \ll 1$.

a PDE-ODE system (2/2)

Lemma (a PDE-ODE system for (ξ, z))

If $u(t)$ is an H^1 solution (NLS) with small mass then $(\xi(t), z(t)) \in P_c H^1 \times \mathbb{C}$ solves

$$\begin{cases} (i\partial_t + H)\xi = B[z]\xi + N_1(z, \xi), \\ \dot{z} + i\Omega(|z|)z = N_2(z, \xi), \end{cases}$$

where

$$B[z]f = P_c(|\Phi[z]|^2 f + \Phi[z]^2 \overline{f}) \;\; : \;\; P_c H^1 \to P_c H^1$$

is $\mathbb{R}$-linear operator, and $\Omega : \mathbb{R}_+ \to \mathbb{R}_+$,
$N_1 : \mathbb{C} \times P_c H^1 \to P_c H^1$ and $N_2 : \mathbb{C} \times P_c H^1 \to \mathbb{C}$ are nonlinearities.

Kenig-Merle argument for the radiation part

Further reduction to a single equation

We know the curve $z(t)$ a priori since it is given by $u(t)$.
Hence, one can regard the above system as a single equation for ξ:

$$(i\partial_t + H)\xi = B[z]\xi + N_1(z,\xi), \quad z(t) \text{ is given curve}$$

We apply the Kenig-Merle type argument to obtain the space-time bound of ξ.

- We recast the theorem as a kind of variational problem;
- The failure of the theorem implies the existence of a ghost minimizer to the problem (use a linearized profile decomposition);
- Derive a contradiction from the existence of the ghost minimizer. Fortunately, this part is the same as the radial case since the spatial shift is controlled by $(**)$.