Comprehensive Testing of Linearity against the Smooth Transition Autoregressive Model

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Abstract

This paper examines the null limit distribution of the quasi-likelihood ratio (QLR) statistic that tests linearity condition using the smooth transition autoregressive (STAR) model. We explicitly show that the QLR test statistic weakly converges to a functional of a Gaussian stochastic process under the null of linearity by resolving the issue of twofold identification meaning that Davies's (1977, 1987) identification problem arises in two different ways under the null. We illustrate our theory using the exponential STAR and logistic STAR models and also conduct Monte Carlo simulations. Finally, we test for neglected nonlinearity in the German money demand, growth rates of US unemployment, and German industrial production. These empirical examples also demonstrate that the QLR test statistic complements the linearity test of the Lagrange multiplier test statistic in Teräsvirta (1994).

Key Words: QLR test statistic, STAR model, linearity test, Gaussian process, null limit distribution, nonstandard testing problem.

Subject Classification: C12, C18, C46, C52.

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1 Introduction

It is popular to test linearity of time series against the smooth transition autoregressive (STAR) model as a first step of building nonlinear STAR models. Luukkonen, Saikkonen, and Teräsvirta (1988), Teräsvirta (1994) and Granger and Teräsvirta (1993) among others suggest to test linearity using the Lagrange multiplier (LM) test statistic based upon the STAR model, and their LM test statistics are popularly used for empirical applications.

On the other hand, the LM test statistic does not comprehensively test for the nonlinearity entailed by the STAR model. As we detail below, the STAR model violates the linearity condition in two different ways, and the LM statistic tests for only one of the two violations to avoid Davies's (1977, 1987) identification problem and obtain a straightforward null limit distribution of the LM test statistic. Despite its handy aspect for applications, the LM test statistic does not entirely test for the nonlinearity involved with the STAR model, and the need for comprehensive testing is a natural consequence of using the STAR model. The main goal of the current study is thus to develop a testing procedure that tests for the alternative nonlinearity in two different ways and combines the testing results into a single test statistic. We achieve this goal by explicitly accommodating Davies's (1977, 1987) identification problem and delivering a methodology to obtain the null limit critical values of the test statistic.

This goal is achieved by applying the results in the previous literature to the STAR model. This literature examines testing linearity using the artificial neural network framework. Cho, Ishida, and White (2011, 2014), Cho and Ishida (2012), White and Cho (2012), and Baek, Cho, and Phillips (2015), among others, study testing for neglected nonlinearity using analytic functions and note that the null of linearity can arise in two or three different ways in their model framework as it does in the STAR model. They call this feature the twofold or trifold identification problem and propose a quasi-likelihood ratio (QLR) test statistic for the problem. We transform their approach to the STAR framework and develop a testing procedure that is readily available for applications.

An encouraging feature of using the QLR test statistic arises from the fact that it is an omnibus test statistic against arbitrary nonlinearity. As Stichcombe and White (1998) point out, the model specification statistic is generically comprehensively revealing if it is constructed by using analytic functions and testing their effect on the specified model as is the case for the QLR test statistic. Many STAR models such as the exponential STAR (ESTAR) and logistic STAR (LSTAR) models contain analytic nonlinear components, and they render the QLR test statistic omnibus against neglected nonlinearity.

Despite the parallel structure of the STAR model-based testing to that of the artificial neural network, the analysis of the QLR test statistic needs to be generalized in order to make the QLR test statistic applicable in the STAR framework. As an example, Cho, Ishida, and White (2011, 2014) characterize the null limit distribution of the QLR test statistic as a functional of a univariate Gaussian stochastic process. This limit distribution cannot, however, be simply applied for STAR models, because, as it turns out, a multidimensional Gaussian stochastic process is required for the null limit distribution. The STAR model has a unique feature that prevents the researcher from applying the artificial neural network approach to the STAR model. We shall generalize the approach based upon the artificial neural network to fit the complexity of the STAR model.

The empirical contribution of this paper is to apply the QLR test statistic to real economic data and demonstrate

its usefulness relative to the LM test statistic. By doing so, our aim is to provide evidence that the QLR and LM test statistics are complementary to each other. For this goal, we examine three well-known empirical examples in the literature using the LM statistic: the German money demand, the US unemployment rate, and the annual growth rate of German industrial production that have been previously studied by Lütkepohl, Teräsvirta, and Wolters (1999), van Dijk, Teräsvirta, and Franses (2002), and Teräsvirta (1994), respectively. Using these data sets and extending them, we illustrate the use of the QLR test statistic along with the LM test statistic and find nonlinear aspects in the data that could not be found by the LM or the QLR statistic alone.

The plan of this paper is as follows. In Section 2, we derive the null limit distribution of the QLR test statistic by resolving the twofold identification problem. We do this by generalizing the approach developed for the artificial neural network model. In Section 3, we apply our theory to the ESTAR and LSTAR models and demonstrate its relevance. In this section we also report resuts on Monte Carlo simulations. In particular, we demonstrate how to apply Hansens's (1996) weighted bootstrap to the QLR test statistic. Section 4 contains applications of the QLR test statistic to the German money demand, the US unemployment rate, and the annual growth rate of German industrial production. The performances of the QLR and LM statistics are compared with each other. The detailed proofs of our claims can be found in the Appendix.

Before proceeding, we provide some notation. A function mapping $f : \mathcal{X} \mapsto \mathcal{Y}$ is denoted by $f(\cdot)$, evaluated derivatives such as $f'(x)|_{x=x_*}$ are written simply as $f'(x_*)$. We also let "$a_n \Rightarrow a$ " and "$a_n \overset{\text{a.s.}}{\to} a$" indicate "$a_n$ weakly converges to a" and "a_n almost surely converges to a," respectively. The latter is occasionally denoted as $\lim_{n\to\infty} a_n \overset{\text{a.s.}}{=} a$.

2 Testing Linearity Using the STAR Model

In this section, we review the literature on testing linearity against STAR. We also consider the QLR test statistic and derive its null limit distribution.

2.1 Motivation of Testing Linearity Using the STAR Model

The following STAR model of order p is popularly specified as a prediction model of a time-series data Y_t (*e.g.*, Teräsvirta, 1994; Granger and Teräsvirta, 1993):

$$\mathcal{M}_0 := \{h_0(\,\cdot\,;\pi,\theta,\gamma,c) : (\pi,\theta,\gamma) \in \Pi \times \Theta \times \Gamma \times C\},$$

where $h_0(z_t;\pi,\theta,\gamma,c) := z_t'\pi + f(\widetilde{z}_t'\alpha - c,\gamma)(z_t'\theta)$, $z_t := (1,\widetilde{z}_t')'$ is a $(p+1)\times 1$ vector of regressors with a transition variable $\widetilde{z}_t'\alpha$. Here, $\widetilde{z}_t := (y_{t-1}, y_{t-2}, \ldots, y_{t-p})'$, and $\alpha := (0,\ldots,1,0,\ldots,0)'$ denotes a selection vector chosen by the researcher. The other parameter vectors $\pi := (\pi_0,\pi_1,\ldots,\pi_p)'$ and $\theta := (\theta_0,\theta_1,\ldots,\theta_p)'$ are the mean transition parameters, and γ is used to describe the smooth transition from one extreme regime to the other. Symbols Π, Θ, Γ, and C denote the parameter spaces of π, θ, γ, and c, respectively. The transition function $f(\cdot,\gamma)$

is a nonlinear, continuously differentiable, and uniformly bounded function. Here, we observe that the empirical researcher often flexibly modifies $\mathcal{M}_0$ by removing the constant from z_t or adding other exogenous variables to z_t. As we see in the empirical section, empirical models are often specified using other dummy variables, so that there is an additional exogenous variable in $\widetilde{z}_t$. Despite the fact that z_t may be different from what we consider here, the null limit distribution of the QLR test statistic is obtained similarly to the current study. So, we fix our ideas by proceeding with our discussions with $\mathcal{M}_0$.

The most popular STAR models are the exponential smooth transition autoregressive (ESTAR) and logistic smooth transition autoregressive (LSTAR) models. They are characterized by the exponential and logistic cumulative distribution functions, respectively, and each of them displays different nonlinear patterns:

$$f_E(\widetilde{z}_t'\alpha - c, \gamma) := 1 - \exp(-\gamma(\widetilde{z}_t'\alpha - c)^2) \quad \text{and} \quad f_L(\widetilde{z}_t'\alpha - c, \gamma) := \{1 + \exp(-\gamma(\widetilde{z}_t'\alpha - c)\}^{-1},$$

where $\gamma > 0$ are the nonlinear functional forms exhibited by the ESTAR and LSTAR models, respectively. It is seen from these expressions that the STAR model has a continuum of regimes defined by transition functions obtaining values between 0 to 1. This aspect makes the model appealing for empirical analysis because the presence of multiple regimes are often structural and attributed to the behaviour of economic agents. For more discussion on the STAR model the reader is referred to van Dijk, Teräsvirta, and Franses (2002), Teräsvirta (1994), Granger and Teräsvirta (1993), and Teräsvirta, Tjøstheim, and Granger (2010), among others.

This study focuses on testing linearity against STAR. The ESTAR and LSTAR models are specified by transforming the exponential function that is analytic, so that it is generically comprehensively revealing for model misspecification as pointed out by Stinchcombe and White (1998). Therefore, the estimated parameters in the transition function become statistically significant such that the nonlinear component necessarily reduces the mean squared error of the model, even when the assumed STAR model is misspecified. This fact implies that if the linear model is misspecified, the mean square error obtained from the STAR models becomes smaller than that from the linear model, motivating testing linearity hypothesis by comparing the estimated mean squared errors from the STAR and the linear model nested in the STAR. The QLR test statistic is often motivated this way. This process delivers an omnibus testing procedure for nonlinearity.

Similar arguments can be found in the previous literature. First, Cho, Ishida, and White (2011, 2014) and Baek, Cho, and Phillips (2015) examine testing linearity using both analytic functions and power transformations. They test linearity using the QLR test statistic and demonstrate usefulness of the test by Monte Carlo experiments. We take advantage of this literature and apply the QLR statistic to testing linearity against STAR. We note, however, that in the previous literature the QLR statistic is applied to testing linearity against artificial neural network models. In the STAR case, the nonlinear functions are different from what they are when the alternative is an artificial neural network. Because of this, the QLR test statistic against STAR exhibits power patterns different from those in Cho, Ishida, and White (2011, 2014) and Baek, Cho, and Phillips (2015). Deriving the null limit distribution of the QLR test based against STAR leads to generalizing the corresponding derivations in Cho, Ishida, and White (2011, 2014) and Baek,

Cho, and Phillips (2015).

Luukkonen, Saikkonen, and Teräsvirta (1988), Teräsvirta (1994), Granger and Teräsvirta (1993), and Teräsvirta, Tjøstheim, and Granger (2010), among others, examine the LM statistic of testing linearity against STAR. As we discuss below, the LM test is defined to test one of the two hypotheses that characterize the linearity condition using the STAR model, whereas the QLR test statistic is defined to handle the two hypotheses at the same time. This aspect of the QLR test statistic extends the testing scope aimed by the LM test statistic, and below we illustrate how the QLR and LM test statistics can complement each other using empirical examples.

2.2 DGP and QLR Test Statistic

In order to proceed, we make the following assumptions:

Assumption 1. $\{y_t : t = 1, 2, \ldots\}$ *is a strictly stationary and absolutely regular process defined on the complete probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, with* $\mathbb{E}[|y_t|] < \infty$ *and mixing coefficient* β_τ *such that for some* $\rho > 1$*,* $\sum_{\tau=1}^{\infty} \tau^{1/(\rho-1)}\beta_\tau < \infty$*.* □

Here, the mixing coefficient is defined as $\beta_\tau := \sup_{s\in\mathbb{N}} \mathbb{E}[\sup_{A\in\mathcal{F}_{s+\tau}^{\infty}} |\mathbb{P}(A|\mathcal{F}_{-\infty}^{s}) - \mathbb{P}(A)|]$, where $\mathcal{F}_\tau^s$ is the σ-field generated by $(y_t, \ldots, y_{t+s})$. Many time series satisfy this condition, and the autoregressive process is one of them. It is general enough to cover the stationary time series we are interested in.

We impose the following regular STAR model condition:

Assumption 2. *Let* $f(\widetilde{z}_t'\alpha, \cdot) : \Gamma \mapsto [0, 1]$ *be a non-polynomial analytic function with probability 1. Let* $\Pi \in \mathbb{R}^{p+1}$*,* $\Theta \in \mathbb{R}^{p+1}$*, and* $\Gamma \in \mathbb{R}$ *be non-empty convex and compact sets such that* $0 \in \Gamma$*. Let* $h(z_t; \pi, \theta, \gamma) := z_t'\pi + \{f(\widetilde{z}_t'\alpha, \gamma) - f(\widetilde{z}_t'\alpha, 0)\}(z_t'\theta)$*, and let* $\mathcal{M} := \{h(\cdot\,; \pi, \theta, \gamma) : (\pi, \theta, \gamma) \in \Pi \times \Theta \times \Gamma\}$ *be the model specified for* $\mathbb{E}[y_t|\widetilde{z}_t]$*.* □

Note that $\mathcal{M}$ differs from $\mathcal{M}_0$ in several respects. First, we set $c = 0$ in $\mathcal{M}_0$ as in the regular exponential autoregressive model in Haggan and Ozaki (1981) because the essential property in testing the linearity is that $f(\widetilde{z}_t'\alpha, \cdot)$ is an analytic function. If c is estimated along with the other parameters π and θ, the null limit distribution of the QLR test becomes more complicated than the one of the current study, and this limits its applicability due to its complexity.

The transition function is centered at $f(\widetilde{z}_t'\alpha, 0)$ for analytical convenience. As $f(\widetilde{z}_t'\alpha, 0)$ is constant, the nonlinearity feature of the STAR model is not modified by the centering. For example, we have $f_E(\widetilde{z}_t'\alpha, 0) = 0$ and $f_L(\widetilde{z}_t'\alpha, 0) = 1/2$, and so it will be centered to have value zero. Furthermore, the centering further reduces the dimension of the identification problem. Without this assumption, π_* and θ_* are not separately identified under the linearity hypothesis, where the subscript '$*$' is used to denote the probability limits of the parameter estimators that are defined below, so that another identification problem is introduced. Specifically, if $\mathbb{E}[y_t|z_t]$ is linear with respect to z_t, we can generate a linear function from $h(\cdot; \pi_*, \theta_*, \gamma_*)$ in two different ways by letting $\theta_* = 0$ or $\gamma_* = 0$. Nevertheless, the linearity hypothesis introduces identification problems. If $\theta_* = 0$, $h(\cdot; \pi_*, 0, \gamma_*) = z_t'\pi_*$, so that γ_* is not identified. That is, Davies's (1977, 1987) identification problem arises, and we call this problem type I identification problem. Alternatively, if $\gamma_* = 0$, $h(\cdot; \pi_*, \theta_*, 0) = z_t'\pi_*$, so that θ_* is not identified, leading to another type of Davies's (1977,

1987) identification problem. We call this problem type II identification problem. On the other hand, if the transition function is not centered at $f(\widetilde{z}_t'\alpha, 0)$, letting $\gamma_* = 0$ and $c = 0$ leads to $h_0(z_t; \pi_*, \theta_*, 0, 0) = z_t'(\pi_* + f(\widetilde{z}_t'\alpha, 0)\theta_*)$. This implies that the type II identification problem becomes more complicated as π_* and θ_* are not separately identified. The centering process is a device to make this complication a relatively simple identification problem. In addition to this, the null limit distribution is not modified by this centering, mainly due to the invariance principle. Note that π in $\mathcal{M}_0$ is reparameterized to $\pi - f(\widetilde{z}_t'\alpha, 0)\theta$ in $\mathcal{M}$, so that the QLR test statistic obtained by this reparameterization becomes identical to that before the reparameterization. Without this reparameterization, the null model investigation has to be separately conducted by discerning the parameters unidentified under $\mathcal{H}_{02}$ as in Cho and Phillips (2018). So, we avoid the involved complication by the centering and obtain the null limit distribution of the QLR test statistic efficiently. We also observe that this centering process is often applied in the literature to derive the null limit distribution of the LM test statistic. For example, Teräsvirta (1994) shows that if z_t contains the constant, the null limit distribution of the LM test statistic is not affected by this centering because the centering parameter is merged with other linear components while applying Taylor expansions.

As already mentioned, the STAR model is different from the artificial neural network model. This is mainly because the transition function is multiplied by $z_t'\theta$, whereas in the ANN model it is multiplied by a constant. This means that the null limit distribution of the QLR test be derived under regularity conditions that are different from those needed for the ANN model. The parameters to be estimated are π, θ, and γ, as α is defined by the researcher.

Using Assumption 2, the linearity hypothesis and the alternative are specified as follows:

$$\mathcal{H}_0 : \exists \pi \in \mathbb{R}^{p+1} \text{ such that } \mathbb{P}(\mathbb{E}[y_t|z_t] = z_t'\pi) = 1; \text{ vs. } \mathcal{H}_1 : \forall \pi \in \mathbb{R}^{p+1}, \mathbb{P}(\mathbb{E}[y_t|z_t] = z_t'\pi) < 1.$$

These hypotheses are the same as the ones in Cho, Ishida, and White (2011, 2014) and Baek, Cho, and Phillips (2015). As in the previous literature, the focus is on developing an omnibus test statistic, but now against STAR, and we use the QLR test statistic as a vehicle for reaching this goal. The QLR test statistic is formally defined as

$$QLR_n := n\left(1 - \frac{\widehat{\sigma}_{n,A}^2}{\widehat{\sigma}_{n,0}^2}\right), \tag{1}$$

where

$$\widehat{\sigma}_{n,0}^2 := \min_{\pi} \frac{1}{n}\sum_{t=1}^{n}(y_t - z_t'\pi)^2, \qquad \widehat{\sigma}_{n,A}^2 := \min_{\pi,\theta,\gamma} \frac{1}{n}\sum_{t=1}^{n}\{y_t - z_t'\pi - f_t(\gamma)(z_t'\alpha)\}^2,$$

and $f_t(\gamma) := f(\widetilde{z}_t'\alpha, \gamma) - f(\widetilde{z}_t'\alpha, 0)$.

The main reason to proceed with the QLR statistic is that the null hypothesis contains type I and II identification problems, and this statistic is able to handle them jointly. As described above, the null holds for the following two sub-hypotheses: $\mathcal{H}_{01} : \theta_* = 0$ and $\mathcal{H}_{02} : \gamma_* = 0$. The limit distribution of the QLR test statistic can be derived both under $\mathcal{H}_{01}$ and $\mathcal{H}_{02}$. We call these derivations type I and type II analysis, respectively. The null hypothesis of linearity against STAR has to be properly tested by tackling both $\mathcal{H}_{01}$ and $\mathcal{H}_{02}$ simultaneously, and we shall demonstrate that the QLR test statistic has the capability of doing so in the vein of the approaches in Cho and White (2007), Cho, Ishida,

and White (2011, 2014), Cho and Ishida (2012), Baek, Cho, and Phillips (2015), and Cho, and Phillips (2015).

The aforementioned LM test statistic does not accommodate the twofold identification problem. Luukkonen, Saikkonen, and Teräsvirta (1988), Teräsvirta (1994), and Granger and Teräsvirta (1993) study the null limit distribution of the LM statistic for testing linearity using $\mathcal{H}_{02}$. The main argument for the LM test is that its null limit distribution is chi-squared, which makes the test easily applicable.

2.3 The Null Limit Distribution of the QLR Test

We now derive the null limit distribution of the QLR test following the approach in Cho, Ishida, and White (2011, 2014) and Baek, Cho, and Phillips (2015) and highlight the difference between the STAR-based approach and the ANN-based one.

Due to the twofold identification problem, we divide our discussion into two parts. We first study the limit distributions of the QLR test under $\mathcal{H}_{01}$ and $\mathcal{H}_{02}$ separately, combine them and, finally, obtain the limit distribution under $\mathcal{H}_0$. For this purpose, we let our objective function or quasi-likelihood (QL) function be

$$\mathcal{L}_n(\pi, \theta, \gamma) := -\sum_{t=1}^{n} \{y_t - z_t'\pi - f_t(\gamma)(z_t'\theta)\}^2.$$

The nonlinear least squares (NLS) estimator $(\widehat{\pi}_n, \widehat{\theta}_n, \widehat{\gamma}_n)$ is obtained by maximizing the QL function with respect to (π, θ, γ).

2.3.1 Type I Analysis: Testing $\mathcal{H}_{01} : \theta_* = 0$

In this subsection, we discuss the limit distribution of the QLR test under $\mathcal{H}_{01} : \theta_* = 0$. The problem is that γ_* is not identified under this hypothesis. We obtain the NLS estimator by maximizing the QL function with respect to γ in the final stage for the purpose of testing $\mathcal{H}_{01}$:

$$\mathcal{L}_n^{(1)} := \max_{\gamma} \max_{\theta} \max_{\pi} -\sum_{t=1}^{n} \{y_t - z_t'\pi - f_t(\gamma)(z_t'\theta)\}^2$$

and let $QLR_n^{(1)}$ be the QLR statistic obtained by this optimization process. That is,

$$\mathcal{L}_n^{(1)} := \max_{\gamma \in \Gamma} \{-u'Mu + u'MF(\gamma)Z[Z'F(\gamma)MF(\gamma)Z]^{-1}Z'F(\gamma)Mu\},$$

where $u := [u_1, u_2, \ldots, u_n]'$, $u_t := y_t - \mathbb{E}[y_t|\widetilde{z}_t]$, $Z := [Z_1, Z_2, \ldots, Z_n]'$, $M := I - Z(Z'Z)^{-1}Z'$, $F(\gamma) := \text{diag}[f_1(\gamma), f_2(\gamma), \ldots, f_n(\gamma)]$, and

$$QLR_n^{(1)} := \max_{\gamma \in \Gamma} \frac{1}{\widehat{\sigma}_{n,0}^2} u'MF(\gamma)Z[Z'F(\gamma)MF(\gamma)Z]^{-1}Z'F(\gamma)Mu \tag{2}$$

under $\mathcal{H}_{01}$ using the fact that $y_t = \mathbb{E}[y_t|\widetilde{z}_t] + u_t = z_t'\pi_* + u_t$. Note that the numerator in the right-hand side of (2) is identical to $n(\widehat{\sigma}^2_{n,0} - \widehat{\sigma}^2_{n,A})$ under $\mathcal{H}_{01} : \theta_* = 0$, so that the definition of the QLR test statistic accords with $QLR_n^{(1)}$.

We now derive the limit distribution of $QLR_n^{(1)}$ under $\mathcal{H}_{01}$ similarly to Cho, Ishida, and White (2011, 2014) and Baek, Cho, and Phillips (2015). For this purpose and to guarantee regular behaviour of the null limit distribution, we impose the following conditions:

Assumption 3. *(i)* $\mathbb{E}[u_t|\widetilde{z}_t, u_{t-1}, \widetilde{z}_{t-1}, \ldots] = 0$*; and (ii)* $\mathbb{E}[u_t^2|\widetilde{z}_t, u_{t-1}, \widetilde{z}_{t-1}, \ldots] = \sigma_*^2$. □

Assumption 4. *(i)* $\sup_{\gamma\in\Gamma} |f_t(\gamma)| \leq m_t$*; and (ii)* $\sup_{\gamma\in\Gamma} |(\partial/\partial\gamma) f_t(\gamma)| \leq m_t$. □

Assumption 5. *There exists a sequence of stationary ergodic random variables* m_t *such that* $|u_t| \leq m_t$*,* $|y_t| \leq m_t$ *and for some* $\omega \geq 2(\rho - 1), \mathbb{E}[m_t^{6+3\omega}] < \infty$*, where* ρ *is given in Assumption 1.* □

Assumption 6. *For each* $\gamma \neq 0$*,* $V_1(\gamma)$ *and* $V_2(\gamma)$ *are positive definite, where for each* γ*,* $V_1(\gamma) := \mathbb{E}[u_t^2\widetilde{r}_t(\gamma)\widetilde{r}_t(\gamma)']$ *and* $V_2(\gamma) := \mathbb{E}[\widetilde{r}_t(\gamma)\widetilde{r}_t(\gamma)']$ *with* $\widetilde{r}_t(\gamma) := (f_t(\gamma)z_t', z_t')'$. □

Assumption 3(*i*) implies that the model in Assumption 2 is not dynamically misspecified, and Assumption 3(*ii*) implies that the error is conditionally homoskedastic. Assumption 4 plays an integral role in applying the tightness condition in Doukhan, Massart, and Rio (1995) to the QLR test statistic. The moment condition in Assumption 5 is stronger than those in Cho, Ishida, and White (2011, 2014), and Assumptions 3 and 5 imply that $\mathbb{E}[u_t^6]$ and $\mathbb{E}[y_t^6]$ are finite. The multiplicative component $f_t(\gamma)z_t'\theta$ in the STAR model makes the stronger moment condition necessary in deriving the regular null limit distribution of the QLR test statistic. Assumption 6 is imposed for the invertibility of the limit covariance matrix. This makes our test statistic non-degenerate.

Given these assumptions, we have the following lemma.

Lemma 1. *Given Assumptions 1, 2, 3(i), 4, 5, 6, and* $\mathcal{H}_{01}$*, (i)* $\widehat{\sigma}^2_{n,0} \xrightarrow{\text{a.s.}} \sigma_*^2 := \mathbb{E}[u_t^2]$*; (ii)* $\{n^{-1/2}Z'F(\cdot)Mu,$ $\widehat{\sigma}^2_{n,0}n^{-1}Z'F(\cdot)MF(\cdot)Z\} \Rightarrow \{\mathcal{Z}_1(\cdot),\ A_1(\cdot,\cdot)\}$ *on* Γ *and* $\Gamma\times\Gamma$*, respectively, where* $\mathcal{Z}_1(\cdot)$ *is a continuous Gaussian process with* $\mathbb{E}[\mathcal{Z}_1(\gamma)] = 0$*, and for each* γ *and* $\widetilde{\gamma}$*,* $\mathbb{E}[\mathcal{Z}_1(\gamma)\mathcal{Z}_1(\widetilde{\gamma})'] = B_1(\gamma,\widetilde{\gamma})$*, where* $B_1(\gamma,\widetilde{\gamma}) := \mathbb{E}[u_t^2 f_t^*(\gamma)f_t^*(\widetilde{\gamma})']$ *and* $A_1(\gamma,\widetilde{\gamma}) := \sigma_*^2\mathbb{E}[f_t^*(\gamma)f_t^*(\widetilde{\gamma})']$ *with* $f_t^*(\gamma) = f_t(\gamma)z_t - \mathbb{E}[f_t(\gamma)z_tz_t']\,\mathbb{E}[z_tz_t']^{-1}z_t$*; (iii) if, in addition, Assumption 3(ii) holds,* $B_1(\gamma,\widetilde{\gamma}) = A_1(\gamma,\widetilde{\gamma})$. □

Lemma 1 plays a central role in deriving the null limit distribution of $QLR_n^{(1)}$ and corresponds to lemma 1 of Cho, Ishida, and White (2011). Despite being similar, the two lemmas are not identical. Note that $\mathcal{Z}_1(\cdot)$ is mapped to $\mathbb{R}^{p+1}$, whereas their lemma obtains a univariate Gaussian process. The multidimensional Gaussian process $\mathcal{Z}_1(\cdot)$ distinguishes the STAR model-based testing from the ANN-based approach. The STAR model has a different null limit distribution by this, and the QLR test based upon the STAR model has power over alternatives in different directions from those of the ANN-based approach.

There is a caveat to Lemma 1. It is clear from (2) that the null limit distribution of $QLR_n^{(1)}$ is determined by the limit behaviour under $\mathcal{H}_{01}$ of both $n^{-1/2}Z'F(\cdot)Mu$ and $n^{-1}Z'F(\cdot)MF(\cdot)Z$. Furthermore, $\lim_{\gamma\to 0} Z'F(\gamma)Mu \overset{\text{a.s.}}{=} Z'F(0)Mu = 0$ and $\lim_{\gamma\to 0} Z'F(\gamma)MF(\gamma)Z \overset{\text{a.s.}}{=} Z'F(0)MF(0)Z = 0$ by the definition of $f_t(\cdot)$. This implies that

it is hard to obtain the limit distribution of $QLR_n^{(1)}$ around $\gamma = 0$. We therefore assume for the moment that 0 is not included in Γ. This condition is relaxed when the limit distribution is examined under $\mathcal{H}_0$.

Theorem 1. *Given Assumptions 1, 2, 3(i), 4, 5, 6, and $\mathcal{H}_{01}$, (i) $QLR_n^{(1)} \Rightarrow \sup_{\gamma\in\Gamma(\epsilon)} \mathcal{G}_1(\gamma)'\mathcal{G}_1(\gamma)$, where $\mathcal{G}_1(\cdot)$ is a Gaussian stochastic process such that for each γ,*

$$\mathbb{E}[\mathcal{G}_1(\gamma)] = 0 \quad \textit{and} \quad \mathbb{E}[\mathcal{G}_1(\gamma)\mathcal{G}_1(\widetilde{\gamma})'] = A_1(\gamma,\gamma)^{-1/2}B_1(\gamma,\widetilde{\gamma})A_1(\widetilde{\gamma},\widetilde{\gamma})^{-1/2},$$

where $\Gamma(\epsilon) := \{\gamma \in \Gamma : |\gamma| \geq \epsilon\}$; (ii) if, in addition, Assumption 3 (ii) holds,

$$\mathbb{E}[\mathcal{G}_1(\gamma)\mathcal{G}_1(\widetilde{\gamma})'] = A_1(\gamma,\gamma)^{-1/2}A_1(\gamma,\widetilde{\gamma})A_1(\widetilde{\gamma},\widetilde{\gamma})^{-1/2}. \qquad \square$$

$\Gamma(\epsilon)$ is considered instead of Γ when γ is excluded around zero. As continuous mapping makes proving Theorem 1 trivial, no proof is given.

Theorem 1 implies that $QLR_n^{(1)}$ does not asymptotically follow a chi-squared distribution under $\mathcal{H}_{01}$ as does the LM statistic in Luukkonen, Saikkonen, and Teräsvirta (1988), Teräsvirta (1994), and Granger and Teräsvirta (1993). The difficulty here is that the null limit distribution contains the unidentified nuisance parameter γ. We can overcome this obstacle by applying Hansen's (1996) weighted bootstrap as in Cho, Cheong, and White (2011) and Cho, Ishida, and White (2011, 2014) to the QLR test statistic.

2.3.2 Type II Analysis: Testing $\mathcal{H}_{02} : \gamma_* = 0$

In this subsection, we study the limit distribution under $\mathcal{H}_{02} : \gamma_* = 0$. This is the null hypothesis used in deriving the LM statistic in Luukkonen, Saikkonen, and Teräsvirta (1988), Teräsvirta (1994), and Granger and Teräsvirta (1993). As we know, θ_* is not identified under $\mathcal{H}_{02}$. We therefore maximize the QL function with respect to θ at the final stage: $\mathcal{L}_n^{(2)} := \sup_\theta \sup_\gamma \sup_\pi -\sum_{t=1}^n \{y_t - z_t'\pi - f_t(\gamma)(z_t'\theta)\}^2$, and denote the QLR test defined by this maximization process by $QLR_n^{(2)}$.

Several remarks are in order. First, maximizing the QL with respect to π is relatively simple due to linearity. We let the concentrated QL (CQL) function be $\mathcal{L}_n^{(2)}(\gamma,\theta) := \sup_\pi \mathcal{L}_n(\pi,\theta,\gamma) = -[y - F(\gamma)Z\theta]'M[y - F(\gamma)Z\theta]$, where $y := [y_1, y_2, \ldots, y_n]$. Second, $\mathcal{L}_n^{(2)}(\cdot)$ is not linear with respect to γ, so that the next stage CQL function with respect to γ cannot be analytically derived. We approximate the CQL function with respect to γ around $\gamma_* = 0$ and capture its limit behaviour under $\mathcal{H}_{02}$. The first-order derivative of $\mathcal{L}_n^{(2)}(\gamma,\theta)$ with respect to γ is

$$\left(\frac{d}{d\gamma}\right)\mathcal{L}_n^{(2)}(\gamma,\theta) = 2[y - F(\gamma)Z\theta]'M\frac{\partial F(\gamma)}{\partial\gamma}Z\theta,$$

where

$$\frac{\partial F(\gamma)}{\partial\gamma} := \left(\frac{\partial}{\partial\gamma}\right)(f(\widetilde{z}_1'\alpha,\gamma), ..., f(\widetilde{z}_n'\alpha,\gamma)).$$

For the LSTAR model, $\partial f_L(\widetilde{z}_t'\alpha,\gamma)/\partial\gamma = f_L(\widetilde{z}_t'\alpha,\gamma)(1 - f_L(\widetilde{z}_t'\alpha,\gamma))\widetilde{z}_t'\alpha$ and $\partial F(0)/\partial\gamma = (1/4)(\widetilde{z}_1'\alpha, ..., \widetilde{z}_n'\alpha)'$,

whereas for ESTAR, $\partial f_E(\widetilde{z}_t'\alpha,\gamma)/\partial\gamma = (\widetilde{z}_t'\alpha)^2(1-f_E(\widetilde{z}_t'\alpha,\gamma))$, so $\partial F(0)/\partial\gamma = ((\widetilde{z}_1'\alpha)^2,...,(\widetilde{z}_n'\alpha)^2)'$, implying that we can approximate the CQL function by a second-order approximation. Nevertheless, as Luukkonen, Saikkonen, and Teräsvirta (1988), Teräsvirta (1994), and Cho, Ishida, and White (2011, 2014) point out, the first-order derivative is often zero for many other models, recommending alternatively applying higher-order Taylor's approximations under their model contexts. Cho, Ishida, and White (2014) adopt a sixth-order Taylor expansion, whereas Luukkonen, Saikkonen, and Teräsvirta (1988), Teräsvirta (1994), and Cho, Ishida, and White (2011) use fourth-order Taylor expansions to obtain the null limit distributions of the LM or QLR test statistics under $\mathcal{H}_{02}$. The order of expansion is determined by the functional form of $f(\widetilde{z}_t'\alpha,\cdot)$.

As our model does not have a fixed form of the STAR model, we fix our model scope by letting κ ($\kappa\in\mathbb{N}$) be the smallest order such that the κ-th order partial derivative with respect to γ is different from zero at $\gamma=0$, so that for all $j<\kappa$, $(\partial^j/\partial\gamma^j)\mathcal{L}_n^{(2)}(0,\cdot)\equiv 0$. Then, the CQL function is expanded as

$$\mathcal{L}_n^{(2)}(\gamma,\theta) = \mathcal{L}_n^{(2)}(0,\theta) + \frac{1}{\kappa!}\frac{\partial^\kappa}{\partial\gamma^\kappa}\mathcal{L}_n^{(2)}(0,\theta)\gamma^\kappa + \ldots + \frac{1}{(2\kappa)!}\frac{\partial^{2\kappa}}{\partial\gamma^{2\kappa}}\mathcal{L}_n^{(2)}(0,\theta)\gamma^{2\kappa} + o_{\mathbb{P}}(\gamma^{2\kappa}). \tag{3}$$

Note that for $j=1,2,\ldots,\kappa-1$, $(\partial^j/\partial\gamma^j)\mathcal{L}_n^{(2)}(0,\theta)=0$ by the definition of κ. The partial derivatives in (3) are given in the following lemma:

Lemma 2. *Given Assumption 2, the definition of κ, and $\mathcal{H}_{02}$,*

$$\frac{\partial^j}{\partial\gamma^j}\mathcal{L}_n^{(2)}(0,\theta) = \begin{cases} 2\theta'Z'H_j(0)Mu, & \text{for } \kappa\le j<2\kappa\,; \\ 2\theta'Z'H_{2\kappa}(0)Mu - \binom{2\kappa}{\kappa}\theta'Z'H_\kappa(0)MH_\kappa(0)Z\theta, & \text{for } j=2\kappa, \end{cases}$$

where $H_j(\gamma) := (\partial^j/\partial\gamma^j)F(\gamma)$. □

Using Lemma 2 we can specifically write (3) as

$$\mathcal{L}_n^{(2)}(\gamma,\theta) - \mathcal{L}_n^{(2)}(0,\theta) = \sum_{j=\kappa}^{2\kappa}\frac{2}{j!}\{\theta'Z'H_j(0)Mu\}\gamma^j - \frac{1}{(2\kappa)!}\binom{2\kappa}{\kappa}\theta'Z'H_\kappa(0)MH_\kappa(0)Z\theta\gamma^{2\kappa} + o_{\mathbb{P}}(\gamma^{2\kappa}). \tag{4}$$

To reduce notational clutter, we further let $G_j := [g_{j,1},g_{j,2},\ldots,g_{j,n}]' := MH_j(0)Z$, where $g_{j,t} := h_{j,t}(0)z_t - Z'H_j(0)Z(Z'Z)^{-1}Z'z_t$ and $\varsigma_n := n^{1/2\kappa}\gamma$ with $h_{j,t}(0)$ being the t-th diagonal element of $H_j(0)$. Then, (4) is written as

$$\mathcal{L}_n^{(2)}(\gamma,\theta) - \mathcal{L}_n^{(2)}(0,\theta) = \sum_{j=\kappa}^{2\kappa}\frac{2}{j!\,n^{j/2\kappa}}\{\theta'G_j'u\}\varsigma_n^j - \frac{1}{(2\kappa)!\,n}\binom{2\kappa}{\kappa}\{\theta'G_\kappa'G_\kappa\theta\}\varsigma_n^{2\kappa} + o_{\mathbb{P}}(\gamma^{2\kappa}). \tag{5}$$

We note that if $j=2\kappa$, $n^{-j/2\kappa}G_j'u = O_{\mathbb{P}}(1)$ by applying the central limit theorem. Furthermore, for $j=\kappa+1,\ldots,2\kappa-1$, $n^{-j/(2\kappa)}(\partial^j/\partial\gamma^j)\mathcal{L}_n^{(2)}(\gamma,\theta) = o_{\mathbb{P}}(1)$ and $\theta'G_{2\kappa}u = o_{\mathbb{P}}(n)$ by the ergodic theorem, so that they become asymptotically negligible, implying that the smallest j-th component greater than κ and surviving at the limit becomes the second-final term in the right side of (5) that is obtained by letting $j=2\kappa$. Note that $n^{-1}G_\kappa'G_\kappa = O_{\mathbb{P}}(1)$, if the ergodic theorem applies. Furthermore, the terms with $j>2\kappa$ belong to $o_{\mathbb{P}}(\gamma^{2\kappa})$, so that they are asymptotically

negligible under the null at any rate. Due to this fact, $\mathcal{L}_n^{(2)}(\cdot,\theta)$ is approximated by the 2κ-th degree polynomial function in (5), and we can establish the following lemma by collecting the terms asymptotically surviving under the null:

Lemma 3. *Given Assumptions 1, 2, 7, and $\mathcal{H}_{02}$, $QLR_n^{(2)} = \sup_\theta \overline{QLR}_n^{(2)}(\theta) + o_{\mathbb{P}}(n)$, where for given θ,*

$$\overline{QLR}_n^{(2)}(\theta) := \sup_{\varsigma_n} \frac{1}{\widehat{\sigma}_{n,0}^2}\left\{\frac{2}{\kappa! n^{1/2}}\{\theta' G_\kappa' u\}\varsigma_n^\kappa - \frac{1}{(2\kappa)! n}\binom{2\kappa}{\kappa}\theta' G_\kappa' G_\kappa \theta \varsigma_n^{2\kappa}\right\}$$

and $\widehat{\varsigma}_n^\kappa(\theta)$ denotes the value of ς_n^κ that maximizes the given objective function, so that

$$\widehat{\varsigma}_n^\kappa(\theta) = \begin{cases} W_n(\theta), & \text{if } \kappa \text{ is odd;} \\ \max[0, W_n(\theta)], & \text{if } \kappa \text{ is even,} \end{cases}$$

where

$$W_n(\theta) := \kappa! n^{1/2} \frac{\theta' G_\kappa' u}{\theta' G_\kappa' G_\kappa \theta}. \qquad \square$$

Lemma 3 implies that the functional form of $\overline{QLR}_n^{(2)}(\cdot)$ depends on κ:

$$\overline{QLR}_n^{(2)}(\theta) = \begin{cases} \frac{1}{\widehat{\sigma}_{n,0}^2}\frac{(\theta' G_\kappa' u)^2}{\theta' G_\kappa' G_\kappa \theta}, & \text{if } k \text{ is odd;} \\ \frac{1}{\widehat{\sigma}_{n,0}^2}\max\left[0, \frac{(\theta' G_\kappa' u)^2}{\theta' G_\kappa' G_\kappa \theta}\right], & \text{if } k \text{ is even.} \end{cases}$$

If θ is a scalar as in the previous literature, θ cancels out, so maximization with respect to θ does not matter any longer, see Cho, Ishida, and White (2011, 2014). This implies that $QLR_n^{(2)}$ and $\overline{QLR}_n^{(2)}(\cdot)$ are asymptotically equivalent under $\mathcal{H}_{02}$. On the other hand, if θ is a vector, the asymptotic null distribution of the test statistic has to be determined by further maximizing $\overline{QLR}_n^{(2)}(\cdot)$ with respect to θ.

We now derive the regular asymptotic distribution of QLR test statistic under $\mathcal{H}_{02}$. The following conditions are sufficient for doing it:

Assumption 7. *For each $j = \kappa, \kappa+1, \ldots, 2\kappa$ and $i = 0, 1, \ldots, p$, (i) $\mathbb{E}[|u_t|^8] < \infty$, $\mathbb{E}[|h_{j,t}(0)|^8] < \infty$, and $\mathbb{E}[|z_{t,i}|^4] < \infty$; or (ii) $\mathbb{E}[|u_t|^4] < \infty$, $\mathbb{E}[|h_{j,t}(0)|^8] < \infty$, and $\mathbb{E}[|z_{t,i}|^8] < \infty$.* $\square$

Assumption 8. *$V_3(0)$ and $V_4(0)$ are positive definite, where for each γ, $V_3(\gamma) := \mathbb{E}[u_t^2 \bar{r}_t(\gamma)\bar{r}_t(\gamma)']$ and $V_4(\gamma) := \mathbb{E}[\bar{r}_t(\gamma)\bar{r}_t(\gamma)']$ with $\bar{r}_t(\gamma) := (h_{t,\kappa}(\gamma) z_t', z_t')'$.* $\square$

Note that γ does not play a significant role in Assumption 8 as it does in the previous case, because $\overline{QLR}_n(\cdot)$ has already concentrated the QL function with respect to γ. Given these regularity conditions, the key limit results of the components that constitute $\overline{QLR}_n^{(2)}(\cdot)$ appear in the following lemma:

Lemma 4. *Given Assumptions 1, 2, 3(i), 4, 7, 8, and $\mathcal{H}_{02}$, (i) $n^{-1/2} G_\kappa' u \Rightarrow \mathcal{Z}_2$, where $\mathbb{E}[\mathcal{Z}_2] = 0$ and $\mathbb{E}[\mathcal{Z}_2 \mathcal{Z}_2'] = \mathbb{E}[u_t^2 g_{t,\kappa} g_{t,\kappa}']$; (ii) $n^{-1} G_\kappa' G_\kappa \xrightarrow{\text{a.s.}} A_2$, where $A_2 := \mathbb{E}[g_{t,\kappa} g_{t,\kappa}']$; and (iii) if, additionally, Assumption 3(iii) holds, $\mathbb{E}[u_t^2 g_{t,\kappa} g_{t,\kappa}'] = \sigma_*^2 \mathbb{E}[g_{t,\kappa} g_{t,\kappa}']$.* $\square$

Using this lemma, the following theorem describes the limit distribution of $QLR_n^{(2)}$ under $\mathcal{H}_{02}$.

Theorem 2. *Given Assumptions 1, 2, 3(i), 4, 7, 8, and $\mathcal{H}_{02}$, (i)*

$$QLR_n^{(2)} \Rightarrow \begin{cases} \max_{\theta\in\Theta} \mathcal{G}_2(\theta)^2, & \text{if } k \text{ is odd;} \\ \max_{\theta\in\Theta} \max[0, \mathcal{G}_2(\theta)]^2, & \text{if } k \text{ is even,} \end{cases}$$

where $\mathcal{G}_2(\cdot)$ is a Gaussian stochastic process such that for each θ, $\mathbb{E}[\mathcal{G}_2(\theta)] = 0$ and

$$\mathbb{E}[\mathcal{G}_2(\theta)\mathcal{G}_2(\widetilde{\theta})] = \frac{B_2(\theta,\widetilde{\theta})}{A_2(\theta,\theta)^{1/2}A_2(\widetilde{\theta},\widetilde{\theta})^{1/2}},$$

where $B_2(\theta,\widetilde{\theta}) := \theta'\mathbb{E}[u_t^2 g_{t,\kappa}g_{t,\kappa}']\widetilde{\theta}$ and $A_2(\theta,\widetilde{\theta}) := \sigma_*^2\theta'\mathbb{E}[g_{t,\kappa}g_{t,\kappa}']\widetilde{\theta}$; (ii) if, additionally, Assumption 3(iii) holds,

$$\mathbb{E}[\mathcal{G}_2(\theta)\mathcal{G}_2(\widetilde{\theta})] = \frac{A_2(\theta,\widetilde{\theta})}{A_2(\theta,\theta)^{1/2}A_2(\widetilde{\theta},\widetilde{\theta})^{1/2}}.$$ □

As Theorem 2 trivially follows from Lemma 4 and continuous mapping, its proof is omitted.

Several remarks are in order. First, the covariance kernel of $\mathcal{G}_2(\cdot)$ is bilinear with respect to θ and $\widetilde{\theta}$. This implies that $\mathcal{G}_2(\theta)$ is a linear Gaussian process with respect to θ. Therefore, if $z \sim N(0, \mathbb{E}[u_t^2 g_{t,\kappa}g_{t,\kappa}'])$, $z'\theta$ as a function of θ is distributionally equivalent to $\mathcal{G}_2(\cdot)$. This fact relates the null limit distribution to the chi-squared distribution. Corollary 1 of Cho and White (2018) shows that $\max_{\theta\in\Theta} \mathcal{G}_2(\theta)^2 \overset{\text{d}}{=} \mathcal{X}_{p+1}^2$ if $\mathcal{G}_2(\cdot)$ is a linear Gaussian process and $\mathbb{E}[u_t^2 g_{t,\kappa}g_{t,\kappa}'] = \sigma_*^2\mathbb{E}[g_{t,\kappa}g_{t,\kappa}']$, where $\mathcal{X}_{p+1}^2$ is a chi-squared distribution with $p+1$ degrees of freedom . Second, the chi-squared null limit distributions of the LM test statistics in Luukkonen, Saikkonen, and Teräsvirta (1988), Teräsvirta (1994), and Granger and Teräsvirta (1993) follow from the fact that the LM test statistic is equivalent to the QLR test statistic under $\mathcal{H}_{02}$. Third, the process to obtain the limit distribution of the QLR test under $\mathcal{H}_{02}$ of here is simpler than that of Cho, Ishida, and White (2011) and Cho and Phillips (2018) in which they examine two other identification problems. In our context, if $f_t(\cdot)$ is not defined by centering $f(\cdot,\gamma)$ at $f(\cdot,0)$, π_* and θ_* are not separately identified.

2.3.3 Null Limit Distribution of the QLR Test Statistic under $\mathcal{H}_0$

In this subsection, we derive the limit distribution of the QLR test under $\mathcal{H}_0$ by examining the relationship between $QLR_n^{(1)}$ and $QLR_n^{(2)}$. Specifically, we show that $QLR_n^{(1)} \geq QLR_n^{(2)}$, which means the limit distribution under $\mathcal{H}_0$ equals that of $QLR_n^{(1)}$. Although this idea is the same as the one in Cho, Ishida, and White (2011, 2014), their approach cannot be applied in the current context. This is because the associated Gaussian process $\mathcal{G}_1(\cdot)$ is multidimensional.

The following lemma generalizes the approach in Cho, Ishida, and White (2011, 2014) to STAR models.

Lemma 5. *Let $n(\gamma) := Z'F(\gamma)Mu$ and $D(\gamma) := Z'F(\gamma)MF(\gamma)Z'$ with $n^{(j)}(\gamma) := (\partial^j/\partial\gamma^j)n(\gamma)$, and $D^{(j)}(\gamma) := (\partial^j/\partial\gamma^j)D(\gamma)$. Under Assumptions 1, 2 and 3, (i) for $j < \kappa$, $\lim_{\gamma\to 0} n^{(j)}(\gamma) \overset{\text{a.s.}}{=} 0$ and $\lim_{\gamma\to 0} D^{(j)}(\gamma) \overset{\text{a.s.}}{=} 0$; (ii) $\lim_{\gamma\to 0} n^{(\kappa)}(\gamma) \overset{\text{a.s.}}{=} G_\kappa'u$; and (iii) $\lim_{\gamma\to 0} D^{(\kappa)}(\gamma) \overset{\text{a.s.}}{=} G_\kappa'G_\kappa$.* □

The limit obtained by letting $\gamma \to 0$ under $\mathcal{H}_{01}$ can be compared with that obtained under $\mathcal{H}_{02}$. More specifically, using Lemma 5 and L'Hospital's rule, we obtain

$$\lim_{\gamma\to 0} n(\gamma)' D(\gamma)^{-1} n(\gamma) \overset{\text{a.s.}}{=} \lim_{\gamma\to 0} n^{(\kappa)}(\gamma)' D^{(\kappa)}(\gamma)^{-1} n^{(\kappa)}(\gamma) \overset{\text{a.s.}}{=} u' G_\kappa (G_\kappa' G_\kappa)^{-1} G_\kappa' u.$$

From this, it follows that $QLR_n^{(1)} \geq \sup_\theta \overline{QLR}_n^{(2)}(\theta)$ because

$$QLR_n^{(1)} = \sup_{\gamma\in\Gamma} \frac{1}{\widehat{\sigma}_{n,0}^2} n(\gamma)' D(\gamma)^{-1} n(\gamma) \geq \lim_{\gamma\to 0} \frac{1}{\widehat{\sigma}_{n,0}^2} n(\gamma)' D(\gamma)^{-1} n(\gamma) \overset{\text{a.s.}}{=} \frac{1}{\widehat{\sigma}_{n,0}^2} u' G_\kappa (G_\kappa' G_\kappa)^{-1} G_\kappa' u.$$

Furthermore, $\overline{QLR}_n^{(2)}(\theta)$ is asymptotically equal to $\frac{1}{\widehat{\sigma}_{n,0}^2} u' G_\kappa \theta (\theta' G_\kappa' G_\kappa \theta)^{-1} \theta' G_\kappa' u$. Thus, it follows that $QLR_n^{(1)} \geq \sup_\theta \overline{QLR}_n^{(2)}(\theta) + o_{\mathbb{P}}(1)$, if

$$G_\kappa (G_\kappa' G_\kappa)^{-1} G_\kappa' - G_\kappa \theta (\theta' G_\kappa' G_\kappa \theta)^{-1} \theta' G_\kappa' \tag{6}$$

is positive semidefinite irrespective of θ. To show this we first note that the two terms in (6) are idempotent and symmetric matrices. Therefore, we may apply Exercise 8.58 in Abadir and Magnus (2005, p. 233). Then,

$$\{G_\kappa (G_\kappa' G_\kappa)^{-1} G_\kappa'\}\{G_\kappa \theta (\theta' G_\kappa' G_\kappa \theta)^{-1} \theta' G_\kappa'\} = G_\kappa \theta (\theta' G_\kappa' G_\kappa \theta)^{-1} \theta' G_\kappa'$$

so that (6) is positive semidefinite. This implies

$$QLR_n = \max[QLR_n^{(1)}, QLR_n^{(2)}] + o_{\mathbb{P}}(1) = \max\left[QLR_n^{(1)}, \sup_\theta \overline{QLR}_n^{(2)}(\theta)\right] + o_{\mathbb{P}}(1) = QLR_n^{(1)} + o_{\mathbb{P}}(1).$$

We can thus conclude that if the conditions in Theorems 1 and 2 hold simultaneously, the null limit distribution of the QLR test statistic is derived by combining Theorems 1 and 2. For this purpose, we combine Assumptions 6 and 8 into a new assumption as follows:

Assumption 9. *For each $\gamma \neq 0$, $V_5(\gamma)$ and $V_6(\gamma)$ are positive definite, where for each γ, $V_5(\gamma) := \mathbb{E}[u_t^2 \ddot{r}_t(\gamma)\ddot{r}_t(\gamma)']$ and $V_6(\gamma) := \mathbb{E}[\ddot{r}_t(\gamma)\ddot{r}_t(\gamma)']$ with $\ddot{r}_t(\gamma) := (h_{t,\kappa}(0) z_t', f_t(\gamma) z_t', z_t')'$.* □

The following theorem now yields the limit distribution of the QLR test under $\mathcal{H}_0$.

Theorem 3. *Given Assumptions 1, 2, 3(i), 4, 5, 7, 9, and $\mathcal{H}_0$, (i) $QLR_n \Rightarrow \sup_{\gamma\in\Gamma} \mathcal{G}_1(\gamma)' \mathcal{G}_1(\gamma)$, where $\mathcal{G}_1(\cdot)$ is a Gaussian stochastic process such that for each γ, $\mathbb{E}[\mathcal{G}_1(\gamma)] = 0$ with*

$$\mathbb{E}[\mathcal{G}_1(\gamma)\mathcal{G}_1(\widetilde{\gamma})'] = A_1(\gamma,\gamma)^{-1/2} B_1(\gamma,\widetilde{\gamma}) A_1(\widetilde{\gamma},\widetilde{\gamma})^{-1/2};$$

(ii) if Assumption 3(ii) also holds,

$$\mathbb{E}[\mathcal{G}_1(\gamma)\mathcal{G}_1(\widetilde{\gamma})'] = A_1(\gamma,\gamma)^{-1/2} A_1(\gamma,\widetilde{\gamma}) A_1(\widetilde{\gamma},\widetilde{\gamma})^{-1/2}.$$ □

Theorem 3 immediately follows from Theorems 1 and 2 and from our earlier argument that $QLR_n = QLR_n^{(1)} + o_{\mathbb{P}}(1)$, which is why we do not prove it in the Appendix. Note that the consequence of Theorem 3 is the same as that of Theorem 1, although the null hypothesis is extended to $\mathcal{H}_0$ from $\mathcal{H}_{01}$ by enlarging the parameter space from $\Gamma(\epsilon)$ to Γ. The given null limit distribution is derived as in Cho, Ishida, and White (2011, 2014) and Baek, Cho, and Phillips (2015). Nevertheless, our proofs generalize theirs due to the existence of the multidimensional Gaussian process. Furthermore, this null limit distribution extends the scope of the LM test statistics in Luukkonen, Saikkonen, and Teräsvirta (1988), Teräsvirta (1994), and Granger and Teräsvirta (1993) who only test $\mathcal{H}_{02}$.

3 Monte Carlo Experiments

In this section, we illustrate testing linearity using the ESTAR and LSTAR models and simulate the QLR test statistic to support the statistical theory in Section 2. Hansen's (1996) weighted bootstrap is also applied to enhance the applicability of our methodology.

3.1 Illustration Using the ESTAR Model

To simplify our illustration, we assume that for all $t = 1, 2, \ldots$, $u_t \sim$ IID $N(0, \sigma_*^2)$ and $y_t = \pi_* y_{t-1} + u_t$ with $\pi_* = 0.5$. Under this DGP, we specify the following first-order ESTAR model:

$$\mathcal{M}_{ESTAR} := \{\pi y_{t-1} + \theta y_{t-1}\{1 - \exp[-\gamma(y_{t-1} - c)^2]\} : \pi \in \Pi,\ \theta \in \Theta,\ \text{and}\ \gamma \in \Gamma\}.$$

The model does not contain an intercept, the transition variable is y_{t-1}, and in what follows we assume that $c = 0$ to avoid unintended identification problem under the null. The nonlinear function $f_t(\gamma) = 1 - \exp(-\gamma y_{t-1}^2)$ is defined on Γ which is a compact and convex, and the exponential function is analytic. This means that the QLR test statistic is generically comprehensively revealing. To identify the model it is assumed that $\gamma_* > 0$, so the identification problem can be avoided. In our model set-up, we allow that 0 is included in Γ. The nonlinear function $f_t(\cdot)$ satisfies $f_t(0) = 0$. Given this model, the following hypotheses are of interest:

$$\mathcal{H}_0' : \exists \pi \in \mathbb{R}, \mathbb{P}(\mathbb{E}[y_t|y_{t-1}] = \pi y_{t-1}) = 1; \ \text{ vs. } \ \mathcal{H}_1' : \forall \pi \in \mathbb{R}, \mathbb{P}(\mathbb{E}[y_t|y_{t-1}] = \pi y_{t-1}) < 1,$$

Two parameter restrictions make $\mathcal{H}_0'$ valid: either $\theta_* = 0$ or $\gamma_* = 0$. The sub-hypotheses are thus $\mathcal{H}_{01}' : \theta_* = 0$ and $\mathcal{H}_{02}' : \gamma_* = 0$.

We first examine the null distribution of the QLR test under $\mathcal{H}_{01}'$. By Theorem 1, the limit null distribution of this test statistic is given as

$$QLR_n^{(1)} = \sup_{\gamma \in \Gamma} \frac{1}{\widehat{\sigma}_{n,0}^2} \frac{(u' M F(\gamma) Z)^2}{Z' F(\gamma) M F(\gamma) Z} \Rightarrow \sup_{\gamma \in \Gamma} \widetilde{\mathcal{G}}_1(\gamma)^2$$

where $\widetilde{\mathcal{G}}_1(\cdot)$ is a mean-zero Gaussian process with the covariance structure

$$\widetilde{\rho}_1(\gamma,\widetilde{\gamma}) = \frac{\widetilde{k}_1(\gamma,\widetilde{\gamma})}{c_1(\gamma,\gamma)^{1/2}\widetilde{c}_1(\widetilde{\gamma},\widetilde{\gamma})^{1/2}} \tag{7}$$

with

$$\widetilde{k}_1(\gamma,\widetilde{\gamma}) = \widetilde{c}_1(\gamma,\widetilde{\gamma}) = \sigma_*^2\mathbb{E}[y_t^2\exp(-(\gamma+\widetilde{\gamma})y_t^2)] - \sigma_*^2\mathbb{E}[y_t^2\exp(-\gamma y_t^2)]\mathbb{E}[y_t^2]^{-1}\mathbb{E}[y_t^2\exp(-\widetilde{\gamma}y_t^2)].$$

Furthermore, under $\mathcal{H}_{01}'$, y_t is normally distributed with $\mathbb{E}[y_t] = 0$ and $\mathrm{var}[y_t] = \sigma_y^2 := \sigma_*^2/(1-\pi_*^2)$, so that y_t^2 follows the gamma distribution with shape parameter $1/2$ and scale parameter $2\sigma_*^2/(1-\pi_*^2)$. Define

$$\widetilde{m}(\gamma) := \left(1 + \frac{2\sigma_*^2}{1-\pi_*^2}\gamma\right)^{-\frac{1}{2}},$$

and

$$\widetilde{h}(\gamma,\widetilde{\gamma}) := \frac{1}{\sigma_y^2}\left(\left[\frac{1+2\sigma_y^2(\gamma+\widetilde{\gamma})}{(1+2\sigma_y^2\gamma)(1+2\sigma_y^2\widetilde{\gamma})}\right]^{-3/2} - 1\right).$$

Note that $\widetilde{m}(\gamma) = \mathbb{E}[\exp(-\gamma y_t^2)]$, so that $\mathbb{E}[y_t^2\exp(-\gamma y_t^2)] = -\widetilde{m}'(\gamma)$. As a result, (7) is further simplified to $\widetilde{k}_1(\gamma,\widetilde{\gamma}) = \sigma_*^2\widetilde{m}'(\gamma)\widetilde{m}'(\widetilde{\gamma})\widetilde{h}(\gamma,\widetilde{\gamma})$, and

$$\widetilde{\rho}_1(\gamma,\widetilde{\gamma}) = \frac{\widetilde{k}_1(\gamma,\widetilde{\gamma})}{\widetilde{c}_1(\gamma,\gamma)^{1/2}\widetilde{c}_1(\widetilde{\gamma},\widetilde{\gamma})^{1/2}} = \frac{\widetilde{h}(\gamma,\widetilde{\gamma})}{\widetilde{h}(\gamma,\gamma)^{1/2}\widetilde{h}(\widetilde{\gamma},\widetilde{\gamma})^{1/2}}.$$

We next examine the limit distribution of the QLR test statistic under $\mathcal{H}_{02}'$: $\gamma_* = 0$. The first-order derivative $(\partial/\partial\gamma)f_t(\gamma) = y_{t-1}^2\exp(-\gamma y_{t-1}^2)$, which is different from zero even when $\gamma = 0$, so that in this case $\kappa = 1$. Thus, we can apply the second-order Taylor expansion to obtain the limit distribution of the QLR test statistic under $\mathcal{H}_{02}'$. As a result,

$$\overline{QLR}_n^{(2)}(\theta) = \frac{1}{\widehat{\sigma}_{n,0}^2}\frac{(\theta' G_\kappa' u)^2}{\theta' G_\kappa' G_\kappa \theta}, \tag{8}$$

where

$$\theta' G_\kappa' u = \theta\left[\sum y_{t-1}^3 u_t - \frac{\sum y_{t-1}^4 \sum y_{t-1}u_t}{\sum y_{t-1}^2}\right] \quad \text{and} \quad \theta' G_\kappa' G_\kappa \theta = \theta^2\left[\sum y_{t-1}^6 - \frac{(\sum y_{t-1}^4)^2}{\sum y_{t-1}^2}\right].$$

In (8), θ is a scalar, so that cancels out, and it follows that $QLR_n^{(2)} \Rightarrow \widetilde{\mathcal{G}}_2^2$, where $\widetilde{\mathcal{G}}_2 \sim N(0,1)$.

These two separate results can be combined, which means that we can examine the limit distribution of the QLR test under $\mathcal{H}_0'$. We have $QLR_n \Rightarrow \sup_{\gamma\in\Gamma}\widetilde{\mathcal{G}}(\gamma)^2$, where

$$\widetilde{\mathcal{G}}(\gamma) = \begin{cases} \widetilde{\mathcal{G}}_1(\gamma), & \text{if } \gamma \neq 0; \\ \widetilde{\mathcal{G}}_2 & \text{otherwise,} \end{cases}$$

and

$$\mathbb{E}[\widetilde{\mathcal{G}}(\gamma)\widetilde{\mathcal{G}}(\widetilde{\gamma})] = \begin{cases} \widetilde{\rho}_1(\gamma,\widetilde{\gamma}), & \text{if } \gamma \neq 0, \widetilde{\gamma} \neq 0; \\ 1 & \text{if } \gamma = 0, \widetilde{\gamma} = 0; \\ \widetilde{\rho}_3(\gamma). & \text{if } \gamma \neq 0, \widetilde{\gamma} = 0, \end{cases}$$

with

$$\widetilde{\rho}_3(\gamma) := \mathbb{E}[\widetilde{\mathcal{G}}_1(\gamma)\widetilde{\mathcal{G}}_2] = \frac{\sqrt{6}\sigma_y^2\gamma}{\widetilde{h}(\gamma,\gamma)^{1/2}(1+2\sigma_y^2\gamma)}$$

such that

$$\widetilde{\rho}_3(\gamma)^2 = \lim_{\widetilde{\gamma}\to 0}\widetilde{\rho}_1(\gamma,\widetilde{\gamma})^2 = \left(\frac{\sqrt{6}\sigma_y^2\gamma}{\widetilde{h}(\gamma,\gamma)^{1/2}(1+2\sigma_y^2\gamma)}\right)^2.$$

Thus, we conclude that $QLR_n \Rightarrow \sup_\gamma \widetilde{\mathcal{G}}(\gamma)^2$, which agrees with Theorem 3.

The null limit distribution can be approximated numerically by simulating a distributionally equivalent Gaussian process. To do this we present the following lemma:

Lemma 6. *If $\{z_k : k = 0, 1, 2, \ldots\}$ is an IID sequence of standard normal random variables, $\widetilde{\mathcal{G}}(\cdot) \overset{\mathrm{d}}{=} \overline{\mathcal{G}}(\cdot)$, where for each $\gamma \in \Gamma := \{\gamma \in \mathbb{R} : \gamma \geq 0\}$,*

$$\overline{\mathcal{G}}(\gamma) := \sum_{k=1}^{\infty} c(\gamma)\cdot a(\gamma)^k \left[(-1)^k\binom{-3/2}{k}\right]^{1/2} z_k, \quad c(\gamma) := \left\{\sum_{k=1}^{\infty}(-1)^k a(\gamma)^{2k}\binom{-3/2}{k}\right\}^{-1/2},$$

and $a(\gamma) := 2\sigma_y^2\gamma/(1+2\sigma_y^2\gamma)$. □

Note that the term $(-1)^k\binom{-3/2}{k}$ in Lemma 6 is always positive irrespective of k, and for any γ,

$$\lim_{k\to\infty} \operatorname{var}\left[a(\gamma)^k\left((-1)^k\binom{-3/2}{k}\right)^{1/2} z_k\right] = \lim_{k\to\infty} a(\gamma)^{2k}(-1)^k\binom{-3/2}{k} = 0 \tag{9}$$

and

$$\widetilde{h}(\gamma,\gamma) = \sum_{k=1}^{\infty} a(\gamma)^{2k}(-1)^k\binom{-3/2}{k}.$$

Using these facts Lemma 6 shows that for any $\gamma, \widetilde{\gamma} \neq 0$, $\mathbb{E}[\overline{\mathcal{G}}(\gamma)\overline{\mathcal{G}}(\widetilde{\gamma})] = \widetilde{\rho}_1(\gamma,\widetilde{\gamma})$. Here, the non-negative parameter space condition for Γ is necessary for $\overline{\mathcal{G}}(\cdot)$ to be properly defined on Γ. Without this condition, $\overline{\mathcal{G}}(\gamma)$ cannot be properly generated. We note that $\lim_{\gamma\downarrow 0}\overline{\mathcal{G}}(\gamma) \overset{\text{a.s.}}{=} z_1$, so that if we let $z_1 = \overline{\mathcal{G}}_2$,

$$\mathbb{E}[\overline{\mathcal{G}}(\gamma)\overline{\mathcal{G}}_2] = \frac{\sqrt{6}\sigma_y^2\gamma}{\widetilde{h}(\gamma,\gamma)^{1/2}(1+2\sigma_y^2\gamma)} = \widetilde{\rho}_3(\gamma).$$

It follows that the distribution of $\widetilde{\mathcal{G}}(\cdot)$ can be simulated by iteratively generating $\overline{\mathcal{G}}(\cdot)$. In practice,

$$\overline{\mathcal{G}}(\gamma; K) := \frac{\sum_{k=1}^{K} a(\gamma)^k[(-1)^k\binom{-3/2}{k}]^{1/2} z_k}{\sqrt{\sum_{k=1}^{K} a(\gamma)^{2k}(-1)^k\binom{-3/2}{k}}}$$

is generated by choosing K to be sufficiently large . By (9), if this is the case, the difference between the distributions of $\overline{\mathcal{G}}(\cdot)$ and $\overline{\mathcal{G}}(\cdot; K)$ becomes negligible.

We now examine the empirical distributions of the QLR statistic under several different environments. First, we consider four different parameter spaces: $\Gamma_1 = [0, 2]$, $\Gamma_2 = [0, 3]$, $\Gamma_3 = [0, 4]$, and $\Gamma_4 = [0, 5]$. They are selected to examine how the null limit distribution of the QLR test is influenced by the choice of Γ. We obtain the limit distribution by simulating $\sup_{\gamma\in\Gamma} \bar{\mathcal{G}}(\gamma; K)^2$ 5,000 times with $K = 2,000$, where Γ is in turn Γ_1, Γ_2, Γ_3, and Γ_4. Second, we study how the empirical distribution of the QLR test statistic changes with the sample size. We consider the sample sizes $n = 100$, $1,000$, $2,000$, and $5,000$.

Figure 1 summarizes the simulation results and shows that the empirical distribution approaches the null limit distribution under different parameter space conditions. We also provide the estimates of the probability density functions next to the empirical distributions. For every parameter space considered, the empirical rejection rates of the QLR test statistics are most accurate when $n = 2,000$. The empirical rejection rates are closer to the nominal levels when the parameter space is small. This result is significant when $n = 100$: the empirical rejection rates for $\Gamma = \Gamma_1$ are closer to the nominal ones than when $\Gamma = \Gamma_4$. Nonetheless, this difference becomes negligible as the sample size increases. The empirical rejection rates obtained using $n = 2,000$ are already satisfactorily close to the nominal levels, and this result is more or less similar to that from 5,000 observations. This suggests that the theory in Section 2 is effective for the ESTAR model. Considering even larger parameter spaces for γ yielded similar results, so they are not reported here.

3.2 Illustration Using the LSTAR Model

As another illustration, we consider testing against the first-order LSTAR model. We assume that the data-generating process is $y_t = \pi_* y_{t-1} + u_t$ with $\pi_* = 0.5$ and

$$u_t = \begin{cases} \ell_t, & \text{w.p. } 1 - \pi_*^2 ; \\ 0, & \text{w.p. } \pi_*^2 \end{cases}$$

where $\{\ell_t\}_{t=1}^n$ follows the Laplace distribution with mean 0 and variance 2. Under this assumption, y_t follows the same distribution as ℓ_t that makes the algebra associated with the LSTAR model straightforward. For example, the covariance kernel of the Gaussian process associated with the null limit distribution of the QLR test statistic is analytically obtained thanks to this distributional assumption. This DGP is a variation of the exponential autoregressive model in Lawrence and Lewis (1980). Their exponential distribution is replaced by the Laplace distribution to allow y_t to obtain negative values.

Given this DGP, the first-order LSTAR model for $\mathbb{E}[y_t|y_{t-1}, y_{t-2}, \ldots]$ is defined as follows:

$$\mathcal{M}^0_{LSTAR} := \{\pi y_{t-1} + \theta y_{t-1}\{1 + \exp(-\gamma y_{t-1})\}^{-1} : \pi \in \Pi,\ \theta \in \Theta,\ \text{and } \gamma \in \Gamma := [0, \bar{\gamma}]\}.$$

The nonlinear logistic function $\{1 + \exp(-\gamma y_{t-1})\}^{-1}$ contains an exponential function. It is therefore analytic, and

this fact delivers a consistent power for the QLR test statistic. Note, however, that for $\gamma = 0$ the value of the logistic function equals 1/2. This difficulty is avoided by subtracting 1/2 from the logistic function when carrying out the test, viz.,

$$\mathcal{M}_{LSTAR} := \{\pi y_{t-1} + \theta y_{t-1}\{[1 + \exp(-\gamma y_{t-1})]^{-1} - 1/2\} : \pi \in \Pi,\ \theta \in \Theta, \text{ and } \gamma \in \Gamma := [0, \bar{\gamma}]\}.$$

By the invariance principle, this shift does not affect the null limit distribution of the QLR test statistic. We here let $\gamma \geq 0$ so that transition function is bounded. If $\gamma < 0$, the transition function may not be bounded. The null and the alternative hypotheses are identical to those in the ESTAR case.

Before proceeding, note that

$$\{1 + \exp(-\gamma y_{t-1})\}^{-1} - \frac{1}{2} = \frac{1}{2}\tanh\left(\frac{\gamma y_{t-1}}{2}\right)$$

Using the hyperbolic tangent function as in Bacon and Watts (1971) makes it easy to find a Gaussian process that is distributionally equivalent to the Gaussian process obtained under the null.

Using this fact, the limit distribution of QLR test statistic under $\mathcal{H}'_{01}$ is derived as in before. By Theorem 1,

$$QLR_n^{(1)} = \sup_{\gamma\in\Gamma} \frac{1}{\widehat{\sigma}^2_{n,0}} \frac{(u'MF(\gamma)Z)^2}{Z'F(\gamma)MF(\gamma)Z} \Rightarrow \sup_{\gamma\in\Gamma} \ddot{\mathcal{G}}_1(\gamma)^2$$

where $\ddot{\mathcal{G}}_1(\cdot)$ is a mean-zero Gaussian process with the covariance structure

$$\ddot{\rho}_1(\gamma, \widetilde{\gamma}) := \frac{\ddot{k}_1(\gamma, \widetilde{\gamma})}{\ddot{c}_1(\gamma, \gamma)^{1/2}\ddot{c}_1(\widetilde{\gamma}, \widetilde{\gamma})^{1/2}}.$$

The function $\ddot{k}_1(\gamma, \tilde{\gamma})$ is equivalent to $\ddot{c}_1(\gamma, \tilde{\gamma})$ by the conditional homoskedasticity condition, and for each nonzero γ and $\widetilde{\gamma}$, we now obtain that

$$\begin{aligned}\ddot{k}_1(\gamma, \tilde{\gamma}) = \frac{1}{4}\mathbb{E}\left[y_{t-1}^2 \tanh\left(\frac{\gamma y_{t-1}}{2}\right)\tanh\left(\frac{\widetilde{\gamma} y_{t-1}}{2}\right)\right] \\ - \frac{1}{4}\mathbb{E}\left[y_{t-1}^2 \tanh\left(\frac{\gamma y_{t-1}}{2}\right)\right]\mathbb{E}[y_{t-1}^2]^{-1}\mathbb{E}\left[y_{t-1}^2 \tanh\left(\frac{\widetilde{\gamma} y_{t-1}}{2}\right)\right].\end{aligned}$$

In the proof of Lemma 7 given in the Appendix, we further show that

$$\ddot{k}_1(\gamma, \widetilde{\gamma}) = \sum_{n=1}^{\infty} b_n(\gamma) b_n(\widetilde{\gamma}),$$

where

$$b_1(\gamma) := \frac{1}{\sqrt{2}}(1 - 2a(\gamma)) \quad \text{with} \quad a(\gamma) := \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(1 + \gamma k)^3}$$

and for $n = 2, 3, \ldots$,

$$b_n(\gamma) := \sqrt{n(n+1)} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(\gamma k)^{n-1}}{(1+\gamma k)^{n+2}}.$$

Next, we derive the limit distribution of the QLR test statistic under $\mathcal{H}'_{02}$. For $\gamma = 0$,

$$(\partial/\partial\gamma) f_t(\gamma) = y_{t-1} \exp(\gamma y_{t-1})/[1 + \exp(-\gamma y_{t-1})]^2 \neq 0,$$

implying that κ is unity as for the ESTAR case, so that we can apply a second-order Taylor expansion to obtain the limit distribution of the QLR test statistic under $\mathcal{H}'_{02}$:

$$\overline{QLR}_n^{(2)}(\theta) = \frac{1}{\widehat{\sigma}^2_{n,0}} \frac{(\theta' G'_\kappa u)^2}{\theta' G'_\kappa G_\kappa \theta},$$

where, similarly to the ESTAR case,

$$\theta' G'_\kappa u = \frac{\theta}{4} \left[\sum y_{t-1}^2 u_t - \frac{\theta}{4} \frac{\sum y_{t-1}^3 \sum y_{t-1} u_t}{\sum y_{t-1}^2} \right] \quad \text{and} \quad \theta' G'_\kappa G_\kappa \theta = \frac{\theta^2}{16} \left[\sum y_{t-1}^4 - \frac{(\sum y_{t-1}^3)^2}{\sum y_{t-1}^2} \right].$$

From this it follows that $QLR_n^{(2)} \Rightarrow \ddot{\mathcal{G}}_2^2$, where $\ddot{\mathcal{G}}_2 \sim N(0,1)$.

Therefore, we conclude that $QLR_n \Rightarrow \sup_\gamma \ddot{\mathcal{G}}(\gamma)^2$, where

$$\ddot{\mathcal{G}}(\gamma) := \begin{cases} \ddot{\mathcal{G}}_1(\gamma), & \text{if } \gamma \neq 0; \\ \ddot{\mathcal{G}}_2, & \text{otherwise.} \end{cases}$$

The limit variance of $\ddot{\mathcal{G}}(\gamma)$ is given as

$$\ddot{\rho}(\gamma, \widetilde{\gamma}) := \mathbb{E}[\ddot{\mathcal{G}}(\gamma)\ddot{\mathcal{G}}(\widetilde{\gamma})] = \begin{cases} \ddot{\rho}_1(\gamma, \widetilde{\gamma}), & \text{if } \gamma \neq 0, \widetilde{\gamma} \neq 0; \\ 1 & \text{if } \gamma = 0, \widetilde{\gamma} = 0; \\ \ddot{\rho}_3(\gamma), & \text{if } \gamma \neq 0, \widetilde{\gamma} = 0, \end{cases}$$

where

$$\ddot{\rho}_3(\gamma) := \mathbb{E}[\ddot{\mathcal{G}}(\gamma)\ddot{\mathcal{G}}_2] = \frac{\ddot{r}_1(\gamma)}{\ddot{k}_1(\gamma,\gamma)^{1/2} \ddot{q}^{1/2}} \quad \text{with} \quad \ddot{r}_1(\gamma) := \frac{1}{2}\mathbb{E}\left[y_{t-1}^3 \tanh\left(\frac{\gamma y_{t-1}}{2}\right)\right] \quad \text{and} \quad \ddot{q} := \mathbb{E}[y_t^4] - \frac{\mathbb{E}[y_t^3]^2}{\mathbb{E}[y_t^2]}.$$

From this it follows that $QLR_n \Rightarrow \sup_{\gamma \in \Gamma} \ddot{\mathcal{G}}(\gamma)^2$. Furthermore, $\mathbb{E}[y_t^3] = 0$ and $\mathbb{E}[y_t^4] = 24$ given our DGP, so that

$$\ddot{\rho}_3(\gamma) = \frac{\mathbb{E}[y_t^3 \tanh(\gamma y_t/2)]}{4\sqrt{6}\ddot{k}_1(\gamma,\gamma)^{1/2}}. \tag{10}$$

Here, we note that

$$\mathbb{E}[y_t^3 \tanh(\gamma y_t/2)] = \frac{1}{8\gamma^4}\left[48\gamma^4 + P_G\left(3, 1 + \frac{1}{2\gamma}\right) - P_G\left(3, \frac{1+\gamma}{2\gamma}\right)\right] \tag{11}$$

by some tedious algebra assisted by Mathematica, where $P_G(n,x)$ is the polygamma function:

$$P_G(n,x) := (d^{n+1}/dx^{n+1})\log(\Gamma(x)).$$

Inserting (11) into (10) yields

$$\ddot{\rho}_3(\gamma) = \frac{1}{32\sqrt{6}\gamma^4 \ddot{k}_1(\gamma,\gamma)^{1/2}} \left[48\gamma^4 + P_G\left(3, 1+\frac{1}{2\gamma}\right) - P_G\left(3, \frac{1+\gamma}{2\gamma}\right)\right]. \tag{12}$$

In addition, we show in the supplementary lemma (Lemma 8) given in the Appendix that applying L'hopital's rule iteratively yields that

$$\lim_{\widetilde{\gamma}\downarrow 0} \ddot{\rho}_1(\gamma,\widetilde{\gamma})^2 = \left(\frac{1}{32\sqrt{6}\gamma^4 \ddot{k}_1(\gamma,\gamma)^{1/2}} \left[48\gamma^4 + P_G\left(3, 1+\frac{1}{2\gamma}\right) - P_G\left(3, \frac{1+\gamma}{2\gamma}\right)\right]\right)^2. \tag{13}$$

This fact implies that $\text{plim}_{\gamma\downarrow 0}\ddot{\mathcal{G}}_1(\gamma)^2 = \ddot{\mathcal{G}}_2^2$. That is, the weak limit of the QLR test statistic under $\mathcal{H}'_{02}$ can be obtained from $\ddot{\mathcal{G}}_1(\cdot)^2$ by letting γ converging to zero, so that $QLR_n \Rightarrow \sup_{\gamma\in\Gamma} \ddot{\mathcal{G}}_1(\gamma)^2$ under $\mathcal{H}'_0$.

Next, we derive another Gaussian process that is distributionally equivalent to $\ddot{\mathcal{G}}(\cdot)$ and conduct Monte Carlo simulations using it. The process is presented in the following lemma.

Lemma 7. *If $\{z_k\}_{k=1}^{\infty}$ is an IID sequences of standard normal random variables, then for each γ and $\widetilde{\gamma} \in \Gamma := \{\gamma \in \mathbb{R} : \gamma \geq 0\}$, $\ddot{\mathcal{G}}(\cdot) \overset{\text{d}}{=} \dot{\mathcal{G}}(\cdot)$, where*

$$\ddot{\mathcal{Z}}_1(\gamma) := \sum_{n=1}^{\infty} b_n(\gamma) z_n \quad \textit{and} \quad \dot{\mathcal{G}}(\gamma) := \left(\sum_{n=1}^{\infty} b_n(\gamma)^2\right)^{-1/2} \ddot{\mathcal{Z}}_1(\gamma).$$

We prove Lemma 7 by showing that the Gaussian process $\dot{\mathcal{G}}(\cdot)$ given in Lemma 7 has the same covariance structure as $\ddot{\mathcal{G}}(\cdot)$, and for this purpose, we focus on proving that for all $\gamma, \widetilde{\gamma} \geq 0$, $\mathbb{E}[\ddot{\mathcal{G}}(\gamma)\ddot{\mathcal{G}}(\widetilde{\gamma})] = \mathbb{E}[\dot{\mathcal{G}}(\gamma)\dot{\mathcal{G}}(\widetilde{\gamma})]$ in the Appendix. If $\gamma, \widetilde{\gamma} > 0$, the desired equality trivially follows from the definition of $\dot{\mathcal{G}}(\cdot)$. On the other hand, applying L'hopital's rule iterative shows that

$$\text{plim}_{\gamma\downarrow 0}\dot{\mathcal{G}}(\gamma) = \frac{\sqrt{3}}{2} z_1 + \frac{1}{2} z_2 \sim N(0,1),$$

so that if we let $\dot{\mathcal{G}}_2 := \lim_{\gamma\downarrow 0} \dot{\mathcal{G}}(\gamma)$, then for $\gamma \neq 0$,

$$\mathbb{E}[\dot{\mathcal{G}}(\gamma)\dot{\mathcal{G}}_2] = \frac{1}{2\ddot{k}_1(\gamma,\gamma)^{1/2}} \left[\sqrt{3} b_1(\gamma) + b_2(\gamma)\right].$$

We show in the proof of Lemma 7 that the term on the right side is identical to $\ddot{\rho}_3(\gamma)$ in (12), so that the covariance kernel of $\dot{\mathcal{G}}(\cdot)$ is identical to $\ddot{\rho}(\cdot,\cdot)$. This fact implies that $\ddot{\mathcal{G}}(\cdot)$ has the same distribution as $\dot{\mathcal{G}}(\cdot)$, and $\dot{\mathcal{G}}_2$ can be regarded as the weak limit obtained under $\mathcal{H}'_{02}$.

Lemma 7 can be used to obtain the approximate null limit distribution of the QLR test statistic. We cannot generate $\dot{\mathcal{G}}(\cdot)$ using the infinite number of $b_n(\cdot)$, but we can simulate the following process to approximate the distribution of

$\dot{\mathcal{G}}(\cdot)$:

$$\dot{\mathcal{G}}(\gamma; K) := \left(\sum_{n=1}^{K} b_{K,n}(\gamma)^2 \right)^{-1/2} \sum_{n=1}^{K} b_{K,n}(\gamma) z_k,$$

where for $n = 2, 3, \ldots,$

$$b_{K,1}(\gamma) := \frac{1}{\sqrt{2}}(1 - 2a_K(\gamma)), \quad a_K(\gamma) := \sum_{k=1}^{K} \frac{(-1)^{k-1}}{(1+\gamma k)^3} \quad \text{and} \quad b_{K,n}(\gamma) := \sqrt{n(n+1)} \sum_{k=1}^{K} \frac{(-1)^{k-1}(\gamma k)^{n-1}}{(1+\gamma k)^3}.$$

If K is sufficiently large, the distribution of $\dot{\mathcal{G}}(\cdot; K)$ is close to that of $\dot{\mathcal{G}}(\cdot)$ as can be easily affirmed by simulations.

We conduct Monte Carlo Simulations for the LSTAR case as in the ESTAR case. The only aspect different from the ESTAR case is that the DGP is the one defined in the beginning of this section. Simulation results are summarized into Figure 2. We use the same parameter spaces $\Gamma = \Gamma_i$, $i = 1, ..., 4$, as before, and we can see that the empirical distribution and PDF estimate of the QLR test approach the null limit distribution and its PDF that are obtained using $\dot{\mathcal{G}}(\cdot; K)$ with $K = 2,500$. This shows that the theory in Section 2 is also valid for the LSTAR model. When the parameter space Γ for γ becomes even larger, we obtain similar results. To save space, they are not reported.

3.3 Application of the Weighted Bootstrap

The standard approach to obtaining the null limit distribution of the QLR test is not applicable for empirical analysis because it requires knowledge of the error distribution. Without this information it is not possible in practice to obtain a distributionally equivalent Gaussian process. Hansen's (1996) weighted bootstrap is useful for this case. We apply it to our models as in Cho and White (2010), Cho, Ishida, and White (2011, 2014), and Cho, Cheong, and White (2011).

Although the relevant weighted bootstrap is available in Cho, Cheong, and White (2011), we provide here a version adapted to the structure of the STAR model. We consider the previously studied ESTAR and LSTAR models and proceed as follows. First, we compute the following score for each grid point of $\gamma \in \Gamma$:

$$\widetilde{W}_n(\gamma) := \frac{1}{n}\sum_{t=1}^{n} \widetilde{u}_{n,t}^2 f_t^2(\gamma) z_t z_t' - \frac{1}{n}\sum_{t=1}^{n} \widetilde{u}_{n,t}^2 f_t(\gamma) z_t z_t' \left[\frac{1}{n}\sum_{t=1}^{n} \widetilde{u}_{n,t}^2 z_t z_t' \right]^{-1} \frac{1}{n}\sum_{t=1}^{n} \widetilde{u}_{n,t}^2 f_t(\gamma) z_t z_t',$$

$$\widetilde{d}_{n,t}(\gamma) := z_t f_t(\gamma) \widetilde{u}_{n,t} - \frac{1}{n}\sum_{t=1}^{n} \widetilde{u}_{n,t}^2 f_t(\gamma) z_t z_t' \left[\frac{1}{n}\sum_{t=1}^{n} \widetilde{u}_{n,t}^2 z_t z_t' \right]^{-1} z_t \widetilde{u}_{n,t},$$

where $\widetilde{u}_{n,t} := y_t - y_{t-1}\widetilde{\theta}_n$, and $\widetilde{\theta}_n$ is the least squares estimator of θ from the null model. Here, $f_t(\gamma) = 1 - \exp(-\gamma y_{t-1}^2)$ for ESTAR and $f_t(\gamma) = \{1 + \exp(\gamma y_{t-1})\}^{-1} - 1/2$ for the LSTAR model. Second, given these functions, we construct the following score function and pseudo-QLR test statistic:

$$\overline{QLR}_{b,n} := \sup_{\gamma \in \Gamma} \left(\frac{1}{\sqrt{n}} \sum_{t=1}^{n} \widetilde{s}_{n,t}(\gamma) \right)' \left(\frac{1}{\sqrt{n}} \sum_{t=1}^{n} \widetilde{s}_{n,t}(\gamma) \right) \quad \text{and} \quad \widetilde{s}_{n,t}(\gamma) := \{\widetilde{W}_n(\gamma)\}^{-1/2} \widetilde{d}_{n,t}(\gamma) z_{b,t},$$

where $z_{b,t} \sim$ IID $N(0,1)$ with respect to b and t, $b = 1, 2, \ldots, B$, and B is the the number of bootstrap replications.

Third, we estimate the empirical p-value by $\widehat{p}_n := B^{-1}\sum_{b=1}^{B}\mathbb{I}[QLR_n < \overline{QLR}_{b,n}]$, where $\mathbb{I}[\cdot]$ is the indicator function. We set $B = 300$ to obtain $\widehat{p}_n^{(i)}$ with $i = 1, 2, \ldots, 2,000$. Finally, for a specified nominal value of α, we compute $\frac{1}{2000}\sum_{i=1}^{2000}\mathbb{I}[\widehat{p}_n^{(i)} < \alpha]$. When the null hypothesis holds, this proportion converges to α.

The results are displayed in the percentile-percentile (PP) plots for the ESTAR and LSTAR models in Figures 3 (ESTAR) and 4 (LSTAR). The horizonal unit interval stands for α, and the vertical unit interval is the space of p-values. As a function of α, the aforementioned proportion should converge to the 45-degree line under the null hypothesis. As before, the four parameter spaces are considered: $\Gamma = \Gamma_i$, $i = 1, ..., 4$. The results are summarized as follows. First, as a function of α, the proportion $\frac{1}{2000}\sum_{i=1}^{2000}\mathbb{I}[\widehat{p}_n^{(i)} < \alpha]$ does converge to the 45-degree line. Second, the empirical rejection rates estimated by the weighted bootstrap are closest to the nominal levels when the parameter space is small. Although the overall finite sample level distortions are smaller for the ESTAR model than the LSTAR model, the empirical rejection rate is close to the nominal significance level if α is close to zero. Finally, as the size of the parameter space increases, more observations are needed to better approximate the 45-degree line in the PP plots. We have conducted simulations using even larger parameter spaces and obtained similar results. We omit reporting them for brevity.

4 Empirical Application

In this section, we illustrate use of the QLR test statistic using three empirical examples and compare the results with those obtained using the LM statistic proposed by Luukkonen, Saikkonen, and Teräsvirta (1988) and applied by Granger and Teräsvirta (1993) and Teräsvirta (1994), among others. This comparison is designed to demonstrate that the QLR and LM tests can complement each other. The p-values of the QLR and LM tests are computed by the weighted bootstrap and the methodology for the F-test statistic in Teräsvirta (1994), respectively.

The three empirical examples in this section have parallel structures. We briefly review the model framework for the LM test statistics. The following auxiliary model is first estimated for the LM test statistics:

$$\mathcal{M}_{AUX} := \{h_{AUX}(\,\cdot\,;\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) : (\alpha_0', \alpha_1', \alpha_1', \alpha_1', \alpha_1')' \in \Theta\},$$

where $h_{AUX}(z_t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) := \alpha_0' z_t + \alpha_1'(\widetilde{z}_t t_t) + \alpha_2'(\widetilde{z}_t t_t^2) + \alpha_3'(\widetilde{z}_t t_t^3) + \alpha_4'(\widetilde{z}_t t_t^4)$, and t_t is the transition variable, viz., $\widetilde{z}_t'\alpha$. This auxiliary model is obtained by applying a fourth-order Taylor's expansion to the analytic function as an intermediate step to compute the LM test statistics conveniently. Although this auxiliary model is different from the STAR model, testing the coefficients of nonlinear components by the LM test statistics turns out to be equivalent to computing the LM test statistics that test the STAR model assumption under $\mathcal{H}_{02}$. Luukkonen, Saikkonen, and Teräsvirta (1988) and Teräsvirta (1994) provide detailed rationales of this equivalence.

We consider the following four sets of hypotheses as common hypotheses of the three empirical examples:

$$\mathcal{H}_{0,1} : \alpha_{1*} = \alpha_{2*} = \alpha_{3*} = 0 | \alpha_{4*} = 0; \text{ vs. } \mathcal{H}_{1,1} : \alpha_{1*} \neq 0, \alpha_{2*} \neq 0, \text{ or } \alpha_{3*} \neq 0 | \alpha_{4*} = 0.$$

$$\mathcal{H}_{0,2} : \alpha_{1*} = \alpha_{2*} = \alpha_{3*} = \alpha_{4*} = 0; \text{ vs. } \mathcal{H}_{1,2} : \alpha_{1*} \neq 0, \alpha_{2*} \neq 0, \alpha_{3*} \neq 0, \text{ or } \alpha_{4*} \neq 0.$$

$$\mathcal{H}_{0,3} : \alpha_{1*} = \alpha_{3*} = 0; \text{ vs. } \mathcal{H}_{1,3} : \alpha_{1*} \neq 0 \text{ or } \alpha_{3*} \neq 0.$$

$$\mathcal{H}_{0,4} : \alpha_{2*} = \alpha_{4*} = 0; \text{ vs. } \mathcal{H}_{1,4} : \alpha_{2*} \neq 0 \text{ or } \alpha_{4*} \neq 0.$$

These hypotheses are specified by following Teräsvirta (1994) and Escribano and Jordà (1999). We denote the LM test statistics testing $\mathcal{H}_{0,i}$ as $LM_{i,n}$, $i = 1, ..., 4$. $LM_{1,n}$ and $LM_{2,n}$ are general tests against STAR. On the other hand, $LM_{3,n}$ and $LM_{4,n}$ are tests against the LSTAR and ESTAR models, respectively. The QLR statistic against ESTAR is denoted by QLR_n^E the one against LSTAR is called QLR_n^L.

4.1 Example 1: German Money Demand

We first consider the German money demand function as examined by Lütkepohl, Teräsvirta, and Wolters (1999). The authors were interested in the stability of this function from 1960Q1 to 1996Q4 and possible nonlinearity. Given that German unification in July 1990 might have affected the money demand function in addition to major monetary policy changes in 1970s, they wanted to check whether these events brought structural changes for German money demand function that could be captured by a nonlinear function. Using LM statistics, they tested for nonlinearity and concluded that the money demand was stable and linear over the period of observation.

We revisit the issue using the data provided by the authors.[1] Following their approach we first let y_t and z_t be Δm_t and $[1, \Delta g_t, \Delta g_{t-1}, \Delta g_{t-2}, \Delta r_t, \Delta r_{t-1}, \Delta p_t, \Delta p_{t-1}, z^*_{t-1}]'$, respectively, where m_t is the log of per capita money stock (M1), g_t is the log of real per capita gross national product (GNP), p_t is the log of the GNP deflator with 1986 as the base-year, and r_t is the long-term interest rate. Here, z^*_t is the co-integration residual obtained by estimating the error-correction model for integrated variables m_t, g_t, g^*_t and r_t:

$$z^*_t = m_t - \delta_1 g_t - \delta_2 g^*_t - \delta_3 r_t,$$

where $g^*_t := \mathbb{I}(t \geq 1990\text{Q3})g_t$. The step dummy variable $\mathbb{I}(t \geq 1990\text{Q3})$ multiplying g_t is intended to accommodate the effect of incorporating the East German economy and population to the per capita GNP due to the German unification. Their STAR model has the following form:

$$y_t = z_t'\pi + \beta_1 d_{1,t} + \beta_2 d_{2,t} + \beta_3 d_{3,t} + \gamma_1 d^*_{1,t} + \gamma_2 d^*_{2,t} + \gamma_3 d^*_{3,t} + z_t'\theta f(\widetilde{z}_t'\alpha; \gamma) + u_t,$$

where $d_{1,t}$, $d_{2,t}$, and $d_{3,t}$ denote seasonal dummies. The other set of dummies, $d^*_{1,t} := \mathbb{I}(t \geq 1990\text{Q3})d_{1,t}$, $d^*_{2,t} := \mathbb{I}(t \geq 1990\text{Q3})d_{2,t}$, and $d^*_{3,t} := \mathbb{I}(t \geq 1990\text{Q3})d_{3,t}$ are included to account for changes in seasonality due to the unification.

[1] The data are available in the data archive of the *Journal of Applied Econometrics*: <http://qed.econ.queensu.ca/jae/1999-v14.5/lutkepohl-terasvirta-wolters/>. The data are not exactly the same data as used in Lütkepohl, Teräsvirta, and Wolters (1999). They used 1991-based GDP deflator series, whereas the provided data are constructed using the 1986-based GDP deflator. By this, estimated parameters are slightly different, although negligible.

We perform the same LM tests and report the p-values in Table 1. Most p-values are more or less the same as the ones in Lütkepohl, Teräsvirta, and Wolters (1999). The differences are due to the use of a 1986-based GNP deflator, whereas they used one with the base year 1991. Out of the p-values reported in Table 1, only $LM_{1,n}$ is significant at the 5% significance level when the transition variable is Δp_t.

Lütkepohl, Teräsvirta, and Wolters (1999) did not apply the statistics $LM_{2,n}$, $LM_{3,n}$, and $LM_{4,n}$. They generally agree with the results from $LM_{1,n}$ in that $LM_{2,n}$ and $LM_{3,n}$ detect nonlinearity at the 5% level when the transition variable is Δp_t. The fact that the p-value of $LM_{3,n}$ is clearly less than that of $LM_{4,n}$ points towards the LSTAR model. The results from QLR_n^L largely agree with what is obtained using $LM_{1,n}$, $LM_{2,n}$ and $LM_{3,n}$. The only diffeence is that linearity is also rejected at the 5% level when the transition variable is Δg_{t-1}. But then, QLR_n^E testing against ESTAR strongly rejects linearity for three transition variables for which no rejection is found by the LM statistics including $LM_{4,n}$. Interestingly, it yields a high p-value when the transition variable is Δp_t. Thus we can conclude that the QLR tests complement the picture by providing information that the LM tests do not contain.

4.2 Example 2: US Unemployment Rates

Next we study performance of the tests when applied to the monthly US unemployment rate. van Dijk, Teräsvirta, and Franses (2002) tested linearity of this series using observations from June 1968 to December 1999. We use the same dataset and compare their LM test statistics with our QLR test statistics. We also extend the series to August 2015 and perform the same tests. Figure 5 shows the US unemployment rate for the extended period.[2]

van Dijk, Teräsvirta, and Franses (2002) point out that the US unemployment rate is a persistent series with an asymmetric adjustment process and strong seasonality. They specify a STAR model with monthly dummy variables for first differences of the seasonally unadjusted unemployment rate of males aged 20 and over . They test linearity against STAR assuming that the transition variable is a lagged twelve-month difference of the unemployment rate. The alternative (STAR) model has the following form (the lag length has been determined by AIC):

$$\begin{aligned}\Delta y_t = \pi_0 + \pi_1 y_{t-1} + \sum_{p=1}^{15} \pi_{p+2} \Delta y_{t-p} + \sum_{k=1}^{11} \pi_{17+k} d_{t,k} \\ + \left[\theta_0 + \theta_1 y_{t-1} + \sum_{p=1}^{15} \theta_{p+2} \Delta y_{t-p} + \sum_{k=1}^{11} \theta_{17+k} d_{t,k}\right] f(\Delta_{12} y_{t-d}; \gamma) + u_t,\end{aligned}$$

where y_t is the monthly US unemployment rate in question, Δy_t is the first difference of y_t, $f(\cdot,\cdot)$ is a nonlinear transition function, $\Delta_{12} y_t$ is the twelve-month difference of y_t, $d_{t,k}$ is the dummy for month k, and $u_t \sim \text{iid}(0, \sigma^2)$. The twelve-month difference $\Delta_{12} y_{t-d}$ is not included as an explanatory variable in the null (linear) model. The theory in Section 2 can nonetheless be used without modification because a null model including $\Delta_{12} y_{t-d}$ can be thought of having a zero coefficient for this variable. Following van Dijk, Teräsvirta, and Franses (2002), we test linearity by using $\Delta_{12} y_{t-d}$, $d = 1, 2, \ldots, 6$, as the transition variable.

[2]The data set used by van Dijk, Teräsvirta, and Franses (2002) is available at <http://swopec.hhs.se/hastef/abs/hastef0380.htm> that was originally retrieved from the Bureau of Labor Statistics.

Our test results using the same series as van Dijk, Teräsvirta, and Franses (2002) can be found in the top panel of Table 2. Both the LM tests and QLR_n^L reject linearity when $d = 2$, and, besides, $LM_{3,n}$ that has power against LSTAR yields $p = 0.057$ for $d = 2$. The p-values of QLR_n^L, however, lie at or below 0.05 for all six lags, suggesting that at least in this particular case this test is more powerful than the LM tests. The smallest p-value is even here attained for $d = 2$. The results from QLR_n^E are quite different and do not suggest any rejection at customary significance levels. This makes sense as this statistic is designed for ESTAR alternatives, and asymmetry in the unemployment rate is best described by an LSTAR model.

The bottom panel of Table 2 contains the results from the series extended to August 2015.[3] Now there seems to be plenty of evidence of asymmetry: all p-values of $LM_{1,n}$ are rather small. $LM_{3,n}$ also has small values for the first three lags, as has $LM_{2,n}$. The p-values from QLR_n^L are smallest of all, which is in line with the results in the top panel. Even QLR_n^E rejects the null of linearity at the 5% level for $d = 1, 2, 3$. This outcome may be expected because the QLR statistics are omnibus tests and as such respond to any deviation from the null hypothesis as the sample size increases. Note, however, that even $LM_{4,n}$ now yields two p-values ($d = 2, 3$) that lie below 0.05, although the test does not have the omnibus property. The behaviour of the unemployment rate during and after the financial crisis (a quick upswing and slow decrease) has probably contributed to theses results.

4.3 Example 3: German Industrial Production

Finally, we examine the growth rate of the seasonally adjusted logarihmic quarterly German industrial production. Teräsvirta (1994) tested linearity of this series against STAR using the LM test and observations from 1961Q1 to 1986Q1. As in the previous examples, we compare these results with the ones obtained using our QLR test statistics.[4]

Following Teräsvirta (1994) we select the ninth-order autoregressive process as the null model by applying the AIC and let the transition variable be the lagged dependent variable for $d = 1, ..., 9$. The null model is (14) with $\gamma_i = 0$, $i = 0, 1, ..., m$. The p-values of the LM and QLR test statistics can be found in the first panel of Table 3. The p-values of both $LM_{1,n}$ and $LM_{2,n}$ test statistics are less than 5% level when $d = 6$. Furthermore, the p-value of $LM_{3,n}$ is 0.063 for this lag. The corresponding p-values for $LM_{4,n}$ are clearly higher than those from the other LM tests. Thus the evidence points towards the LSTAR model. On the other hand, the evidence against the null hypothesis according to the QLR tests is rather weak. It seems that $LM_{4,n}$ and QLR_n^E both agree that there is no evidence in the series supporting an ESTAR specification. The QLR_n^L statistic seems to offer even less evidence against linearity than $LM_{1,n}$ and $LM_{2,n}$ do.

We next increase the time span to 2015Q1 from the same motivation as in the previous subsection. We specify a model different from the model for 1961Q1 to 1986Q1 to accommodate the Germany reunification effect. Merging the East-German industrial production with the West-German one caused a shift in the series, and this was taken into

[3]The recent observations of the monthly US unemployment rate are available at <http://beta.bls.gov/dataViewer/view/timeseries/LNU04000025>.

[4]The data are collected from the OECD main economic indicators: <https://data.oecd.org/industry/industrial-production.htm>.

account by introducing the step dummy $D_t := \mathbb{I}[t \geq 1990Q3]$. The resulting linear null model has the following form:

$$\Delta y_t = \beta_0 + \sum_{j=1}^{k} \beta_j y_{t-j} + \gamma_0 D_t + \sum_{j=1}^{m} \gamma_j D_t y_{t-j} + u_t, \tag{14}$$

where $y_t = \ln(Y_t)$, and Y_t denotes the original seasonally adjusted time series. The lag lengths $k = 6$ and $m = 2$ are determined by AIC. The transition variable is Δy_{t-d}, $d = 1, ..., 6$. The test results can be found in the bottom panel of Table 3.

Increasing the sample size brings more evidence against the null hypothesis. All LM tests now strongly reject the for the first two lags. The results are ambiguous in the sense that both $LM_{1,n}$ and $LM_{2,n}$ yield strong rejections at these two lags and have a p-value below 0.05 also when $d = 4$. The statistic QLR_n^L rejects the null hypothesis slightly more often than QLR_n^E, but the smallest p-values in both occur at different lags except for $d = 4$. The deep trough in the growth rate around 2008–2009 may have contributed to this ambiguity. Interestingly, the LM tests now provide more information against linearity than the two omnibus statistics QLR_n^L and QLR_n^E. This seems to suggest that the QLR statistics do not necessarily dominate their LM counterparts when it comes to power. Rather, it might be better to view these two types of tests as complementing each other.

5 Conclusion

The current study examines the null limit distribution of the QLR test statistic for neglected nonlinearity using the STAR model. The QLR test statistic contains a twofold identification problem under the null, and we explicitly examine how the twofold identification problem affects the null limit distribution of the QLR test statistic. We show that the QLR test statistic is shown to converge to a functional of a multidimensional Gaussian stochastic process under the null of linearity by extending the testing scope of the LM test statistic in Luukkonen, Saikkonen, and Teräsvirta (1988), Teräsvirta (1994) and Granger and Teräsvirta (1993).

We further illustrate our theory on the QLR test statistic to ESTAR and LSTAR models and affirm our theory by obtaining the null limit critical values and conducting Monte Carlo simulations. Finally, three empirical examples are revisited by comparing the QLR and LM test statistics, and we demonstrate how to complement each test statistic.

6 Appendix

Proof of Lemma 1: (*i*) Given Assumptions 1, 2, 3, and 5, it is trivial to show that $\widehat{\sigma}_{n,0}^2 \xrightarrow{\text{a.s.}} \sigma_*^2$ by the ergodic theorem.

(*ii*) The null limit distribution of $QLR_n^{(1)}$ is determined by the two terms in (2): $Z'F(\cdot)Mu$ and $Z'F(\cdot)MF(\cdot)Z$. We examine their null limit behaviour one by one and combine the limit results using the converging-together lemma in Billingsley (1999, p. 39).

(a) We show the weak convergence part of $n^{-1/2}Z'F(\cdot)Mu$. Using the definition of $M := I - Z(Z'Z)^{-1}Z'$ we have $Z'F(\gamma)Mu = Z'F(\gamma)u - Z'F(\gamma)Z(Z'Z)^{-1}Z'u$, and we now examine the components on the right-hand side

of this equation separately. For each $\gamma \in \Gamma$, we define

$$\widehat{f}_{n,t}(\gamma) := f_t(\gamma)u_t z_t - \left(\sum_{t=1}^n f_t(\gamma) z_t z_t'\right)\left(\sum_{t=1}^n z_t z_t'\right)^{-1} \sum_{t=1}^n z_t u_t,$$

$$\widetilde{f}_{n,t}(\gamma) := f_t(\gamma)u_t z_t - \mathbb{E}[f_t(\gamma) z_t z_t']\mathbb{E}[z_t z_t']^{-1} \sum_{t=1}^n z_t u_t$$

and show that

$$\sup_{\gamma \in \Gamma(\epsilon)} \left\| n^{-1/2} \sum_{t=1}^n \left[\widehat{f}_{n,t}(\gamma) - \widetilde{f}_{n,t}(\gamma)\right] \right\|_\infty = o_{\mathbb{P}(1)}, \tag{15}$$

where $\Gamma(\epsilon) := \{\gamma \in \Gamma : |\gamma| \geq \epsilon\}$ and $\|\cdot\|_\infty$ is the uniform matrix norm. We have

$$\begin{aligned}
&\sup_{\gamma \in \Gamma(\epsilon)} \left\| \frac{1}{\sqrt{n}} \sum_{t=1}^n \left[\widehat{f}_{n,t}(\gamma) - \widetilde{f}_{n,t}(\gamma)\right] \right\|_\infty \\
&\quad \leq \sup_{\gamma \in \Gamma(\epsilon)} \left\| \left(n^{-1} \sum_{t=1}^n f_t(\gamma) z_t z_t'\right) \left\{ \left(n^{-1} \sum_{t=1}^n z_t z_t'\right)^{-1} - \mathbb{E}[z_t z_t']^{-1} \right\} n^{-1/2} \sum_{t=1}^n z_t u_t \right\|_\infty \\
&\qquad + \sup_{\gamma \in \Gamma(\epsilon)} \left\| \left\{ \left(n^{-1} \sum_{t=1}^n f_t(\gamma) z_t z_t'\right) - \mathbb{E}[f_t(\gamma) z_t z_t'] \right\} \mathbb{E}[z_t z_t']^{-1} n^{-1/2} \sum_{t=1}^n z_t u_t \right\|_\infty .
\end{aligned} \tag{16}$$

We show that each term on the right-hand side of (16) is $o_{\mathbb{P}}(1)$. Now, $\{z_t u_t, \mathcal{F}_t\}$ is a martingale difference sequence, where $\mathcal{F}_t$ is the smallest sigma-field generated by $\{z_t u_t, z_{t-1} u_{t-1}, \ldots\}$. Therefore, $\mathbb{E}[z_t u_t | \mathcal{F}_{t-1}] = 0$,

$$\mathbb{E}[|Z_{t,j} u_t|^2] = \mathbb{E}[u_t^4]^{1/2} \mathbb{E}[|Z_{t,j}|^4]^{1/2} \leq \mathbb{E}[m_t^4]^{1/2} \mathbb{E}[Z_{t,j}^4]^{1/2} < \infty,$$

and $\mathbb{E}[u_t^2 z_t z_t']$ is positive definite. Thus, $n^{1/2} \sum_{t=1}^n z_t u_t$ is asymptotically normal. Next, we note that $n^{-1/2} \sum_{t=1}^n f_t(\gamma) u_t z_t$ is also asymptotically normal. This follows from the fact that $\{f_t(\gamma) u_t z_t, \mathcal{F}_t\}$ is a martingale difference sequence, and for each j, $|f_t(\gamma) u_t z_{t,j}|^2 \leq m_t^6$, and $\mathbb{E}[m_t^6] < \infty$ by Assumptions 4 and 5. Furthermore, $\sup_{\gamma \in \Gamma} \left\| n^{-1} \sum_{t=1}^n f_t(\gamma) z_t z_t' - \mathbb{E}[f_t(\gamma) z_t z_t'] \right\|_\infty = o_{\mathbb{P}}(1)$ by Ranga Rao's (1962) uniform law of large numbers. Thus,

$$\sup_{\gamma \in \Gamma(\epsilon)} \left\| \left\{ n^{-1} \sum_{t=1}^n f_t(\gamma) z_t z_t' - \mathbb{E}[f_t(\gamma) z_t z_t'] \right\} \mathbb{E}[z_t z_t']^{-1} n^{-1/2} \sum_{t=1}^n z_t u_t \right\|_\infty = o_{\mathbb{P}}(1). \tag{17}$$

This shows that the second term of (16) is $o_{\mathbb{P}}(1)$. We now demonstrate that the first term of (16) is also $o_{\mathbb{P}}(1)$. By Assumption 4 and the ergodic theorem, we note that

$$\left\| n^{-1} \sum_{t=1}^n z_t z_t' - \mathbb{E}[z_t z_t'] \right\|_\infty = o_{\mathbb{P}}(1), \qquad \left| \sum_{t=1}^n f_t(\gamma) z_{t,j} z_{t,i} \right| \leq \sum_{t=1}^n m_t^3 = O_{\mathbb{P}}(n)$$

so that (17) follows, leading to (15). Therefore, $n^{-1/2} Z' F(\gamma) M u \overset{\text{A}}{\sim} N[0, B_1(\gamma, \gamma)]$ by noting that $\mathbb{E}[\widetilde{f}_{n,t}(\gamma) \widetilde{f}_{n,t}(\gamma)'] =$

$B_1(\gamma,\gamma)$. Using the same methodology, we can show that for each $\gamma, \widetilde{\gamma} \in \Gamma(\epsilon)$,

$$\frac{1}{\sqrt{n}} \begin{bmatrix} Z'F(\gamma)Mu \\ Z'F(\widetilde{\gamma})Mu \end{bmatrix} \overset{\text{A}}{\sim} N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{bmatrix} B_1(\gamma,\gamma) & B_1(\gamma,\widetilde{\gamma}) \\ B_1(\widetilde{\gamma},\gamma) & B_1(\widetilde{\gamma},\widetilde{\gamma}) \end{bmatrix} \right].$$

Finally, we have to show that $\{\widetilde{f}_{n,t}(\cdot)\}$ is tight. First note that by Assumptions 1, 2, and 4, it follows that $|f_t(\gamma)z_{t,j}u_t - f_t(\widetilde{\gamma})z_{t,j}u_t| \leq m_t \left|z_{t,j}u_t\right| \ |\gamma - \widetilde{\gamma}|$ for each j. From this we obtain

$$\sup_{|\gamma-\widetilde{\gamma}|<\eta} |f_t(\gamma)z_{t,j}u_t - f_t(\widetilde{\gamma})z_{t,j}u_t|^{2+\omega} \leq m_t^{2+\omega}|z_{t,j}u_t|^{2+\omega}\eta^{2+\omega} \leq m_t^{6+3\omega}\eta^{2+\omega},$$

so that

$$\mathbb{E}\left[\sup_{|\gamma-\widetilde{\gamma}|<\eta} |f_t(\gamma)z_{t,j}u_t - f_t(\widetilde{\gamma})z_{t,j}u_t|^{2+\omega}\right]^{\frac{1}{2+\omega}} \leq \mathbb{E}[m_t^{6+3\omega}]^{\frac{1}{2+\omega}}\eta.$$

for each j. This implies that $\{n^{-1/2}f_t(\cdot)z_{t,j}u_t\}$ is tight because Ossiander's $L^{2+\omega}$ entropy is finite.

Next, for some $c > 0$,

$$\begin{aligned}
&\left\|\mathbb{E}[f_t(\gamma)z_tz_t']\mathbb{E}[z_tz_t']^{-1}z_tu_t - \mathbb{E}[f_t(\widetilde{\gamma})z_tz_t']\mathbb{E}[z_tz_t']^{-1}z_tu_t\right\|_\infty \\
&= \left\|\mathbb{E}[\{f_t(\gamma) - f_t(\widetilde{\gamma})\}z_tz_t']\mathbb{E}[z_tz_t']^{-1}z_tu_t\right\|_\infty \leq cm_t^2 \left\|\mathbb{E}[z_tz_t']^{-1}\right\|_\infty \left\|\mathbb{E}[\{f_t(\gamma) - f_t(\widetilde{\gamma})\}z_tz_t']\right\|_\infty,
\end{aligned}$$

by the property of the uniform norm and Assumption 5. Also note that $\|\mathbb{E}[f_t(\gamma)z_tz_t' - f_t(\widetilde{\gamma})z_tz_t']\|_\infty \leq \|\mathbb{E}[\{f_t(\gamma) - f_t(\widetilde{\gamma})\}z_tz_t']\|_1$ and by Assumption 4, for each $i, j = 1, 2, \ldots, m+1$, $|z_{t,j}z_{t,i}[f_t(\gamma) - f_t(\widetilde{\gamma})]| \leq m_t^3|\gamma - \widetilde{\gamma}|$, where $\|[g_{i,j}]\|_1 := \sum_i \sum_j |g_{i,j}|$. Therefore,

$$\begin{aligned}
&\left\|\mathbb{E}[f_t(\gamma)z_tz_t']\mathbb{E}[z_tz_t']^{-1}z_tu_t - \mathbb{E}[f_t(\widetilde{\gamma})z_tz_t']\mathbb{E}[z_tz_t']^{-1}z_tu_t\right\|_\infty \\
&\leq cm_t^2 \left\|\mathbb{E}[z_tz_t']^{-1}\right\|_\infty \left\|\mathbb{E}[\{f_t(\gamma) - f_t(\widetilde{\gamma})\}z_tz_t']\right\|_\infty \leq c(m+1)^2m_t^2 \left\|\mathbb{E}[z_tz_t']^{-1}\right\|_\infty \mathbb{E}[m_t^3]|\gamma - \widetilde{\gamma}|. \qquad (18)
\end{aligned}$$

This inequality (18) implies that $\{n^{-1/2}\mathbb{E}[f_t(\cdot)z_tz_t']\mathbb{E}[z_tz_t']^{-1}z_tu_t\}$ is also tight. Hence, it follows that for some $b < \infty$,

$$\mathbb{E}\left[\sup_{|\gamma-\widetilde{\gamma}|<\eta} |\widetilde{f}_t(\gamma) - \widetilde{f}_t(\widetilde{\gamma})|^{2+\omega}\right] \leq b \cdot \eta.$$

That is, $\{n^{-1/2}\sum_{t=1}^n \widetilde{f}_{n,t}(\cdot)\}$ is tight. From this and the fact that the finite-dimensional multivariate CLT holds, the weak convergence of $\{n^{-1/2}\sum_{t=1}^n \widetilde{f}_{n,t}(\cdot)\}$ is established.

(b) Next, we examine the limit behaviour of $n^{-1}Z'F(\cdot)F(\cdot)Z$. For this purpose, we note that

$$\frac{1}{n}Z'F(\gamma)F(\gamma)Z = \frac{1}{n}\sum_{t=1}^n f_t(\gamma)^2z_tz_t' - \left\{\frac{1}{n}\sum_{t=1}^n f_t(\gamma)z_tz_t'\right\}\left\{\frac{1}{n}\sum_{t=1}^n z_tz_t'\right\}^{-1}\left\{\frac{1}{n}\sum_{t=1}^n f_t(\gamma)z_tz_t'\right\},$$

and, given Assumptions 1, 2, 3, 4, and 6, by Ranga Rao's (1962) uniform law of large numbers we have

$$\sup_{\gamma\in\Gamma(\epsilon)}\left\|n^{-1}\sum_{t=1}^{n}f_t(\gamma)^2z_tz_t'-\mathbb{E}[f_t(\gamma)^2z_tz_t']\right\|\xrightarrow{\text{a.s.}}0,\ \sup_{\gamma\in\Gamma(\epsilon)}\left\|n^{-1}\sum_{t=1}^{n}f_t(\gamma)z_tz_t'-\mathbb{E}[f_t(\gamma)z_tz_t']\right\|\xrightarrow{\text{a.s.}}0.$$

Therefore, given $\left\|n^{-1}\sum_{t=1}^{n}z_tz_t'-\mathbb{E}[z_tz_t']\right\|_\infty=o_{\mathbb{P}}(1)$, it follows that

$$\sup_{\gamma\in\Gamma(\epsilon)}\left|n^{-1}Z'F(\gamma)MF(\gamma)Z-\{\mathbb{E}[f_t(\gamma)^2z_tz_t']-\mathbb{E}[f_t(\gamma)z_tz_t']\mathbb{E}[z_tz_t']^{-1}\mathbb{E}[f_t(\gamma)z_tz_t']\}\right|=o_{\mathbb{P}}(1).$$

Applying the converging-together lemma yields the desired result.

(*iii*) This result trivially follows from the fact that $\mathbb{E}[u_t^2|z_t]=\sigma_*^2$. ■

Proof of Lemma 2: Given Assumption 2, $\mathcal{H}_{02}$, and the definition of $H_j(\gamma)$, the j-th order derivative of $\mathcal{L}_n^{(2)}(\,\cdot\,;\theta)$ is obtained as

$$\begin{aligned}\frac{\partial^j}{\partial\gamma^j}\mathcal{L}_n^{(2)}(\gamma;\theta)&=-\sum_{k=0}^{j}\binom{j}{k}\left\{\frac{\partial^k}{\partial\gamma^k}(y-F(\gamma)Z\theta)'\right\}M\left\{\frac{\partial^{j-k}}{\partial\gamma^{j-k}}(y-F(\gamma)Z\theta)\right\}\\&=2\theta'Z'H_j(\gamma)Mu-\sum_{k=1}^{j-1}\binom{j}{k}\theta'Z'H_j(\gamma)MH_{j-k}(\gamma)Z\theta\end{aligned}\tag{19}$$

by iteratively applying the general Leibniz rule. We now evaluate this derivative at $\gamma=0$. Note that $H_j(0)=0$ if $j<\kappa$ by the definition of κ. This implies that $(\partial^j/\partial\gamma^j)\mathcal{L}_n^{(2)}(0;\theta)=0$ for $j=1,2,\ldots,\kappa-1$. This also implies that $\binom{j}{k}\theta'Z'H_j(0)MH_{j-k}(0)Z\theta=0$ for $j=\kappa,\kappa+1,\ldots,2\kappa-1$. Therefore, $\frac{\partial^j}{\partial\gamma^j}\mathcal{L}_n^{(2)}(0;\theta)=2\theta'Z'H_j(0)Mu$. Finally, we examine the case in which $j=2\kappa$. For each $j<2\kappa$, $H_j(0)=0$ and $H_\kappa(0)\neq0$, so that the summand of the second term in the right side of (19) is different from zero only when $j=2\kappa$ and $k=\kappa$:

$$\frac{\partial^{2\kappa}}{\partial\gamma^{2\kappa}}\mathcal{L}_n^{(2)}(0;\theta)=2\theta'Z'H_{2\kappa}(\gamma)Mu-\binom{2\kappa}{\kappa}\theta'Z'H_\kappa(\gamma)MH_\kappa(\gamma)Z\theta.$$

This completes the proof. ■

Proof of Lemma 3: Given Assumptions 1, 2, 7, and $\mathcal{H}_{02}$, we note that

$$QLR_n^{(2)}:=\sup_\theta\overline{QLR}_n^{(2)}(\theta)=\sup_\theta\sup_\varsigma\frac{1}{\widehat{\sigma}_{n,0}^2}\left[\frac{2\{\theta'G_\kappa'u\}\varsigma^\kappa}{\kappa!\sqrt{n}}-\frac{1}{(2\kappa)!n}\left\{\binom{2\kappa}{\kappa}\theta'G_\kappa'G_\kappa\theta\right\}\varsigma^{2\kappa}\right]+o_{\mathbb{P}}(n).\tag{20}$$

Then, the FOC with respect to ς implies that

$$\widehat{\varsigma}_n^\kappa(\theta)=\begin{cases}W_n(\theta), & \text{if } \kappa \text{ is odd;}\\ \max[0,W_n(\theta)] & \text{if } \kappa \text{ is even}\end{cases}$$

by noting that $\widehat{\varsigma}_n^\kappa(\theta)$ cannot be negative. If we plug $\widehat{\varsigma}_n^\kappa(\theta)$ back into the right side of (20), the desired result follows. ■

Proof of Lemma 4: Before proving Lemma 4, we first show that for each j, $Z'H_j(0)Mu=O_{\mathbb{P}}(n^{1/2})$, so that

$j = \kappa + 1, \ldots, 2\kappa - 1$, $Z'H_j(0)Mu = o_{\mathbb{P}}(n^{j/2\kappa})$. Note that for $j = \kappa + 1, \ldots, 2\kappa$,

$$Z'H_jMu = \sum_{t=1}^{n} z_t h_{t,j}(0) u_t - \sum_{t=1}^{n} z_t h_{t,j}(0) z_t' \left(\sum_{t=1}^{n} z_t z_t' \right)^{-1} \sum_{t=1}^{n} z_t u_t.$$

First we apply the ergodic theorem to $n^{-1} \sum_t z_t h_{t,j}(0) z_t'$ and $n^{-1} \sum_t z_t z_t'$, respectively. Second, given Assumptions 1, 2, 3, 7, and 8, following the proof of Lemma 1, we have that $n^{-1/2} \sum_t z_t u_t$ is asymptotically normal. Furthermore, for all $j = \kappa + 1, \ldots, 2\kappa$, we show that $n^{-1/2} \sum_t z_t h_{t,j}(0) u_t$ is asymptotically normal. To do this, first note that $\{z_t h_{t,j}(0) u_t, \mathcal{F}_t\}$ is a martingale difference sequence, so that for each j, $\mathbb{E}[z_t h_{t,j}(0) u_t | \mathcal{F}_{t-1}] = 0$. Next, we prove that for each j, $\mathbb{E}[z_{t,i}^2 h_{t,j}^2(0) u_t^2] < \infty$. First note that using the moment conditions in Assumption 7,

$$\mathbb{E}[|z_{t,i}^2 h_{t,j}^2(0) u_t^2|] \leq \mathbb{E}[|u_t|^4]^{1/2} \mathbb{E}[|h_{t,j}^2(0) z_{t,i}^2|^2]^{1/2} \leq \mathbb{E}[|u_t|^4]^{1/2} \mathbb{E}[|h_{t,j}(0)|^8]^{1/4} \mathbb{E}[|z_{t,i}|^8]^{1/4} < \infty$$

by the Cauchy-Schwarz's inequality. For the same reason,

$$\mathbb{E}[|z_{t,i}^2 h_{t,j}^2(0) u_t^2|] \leq \mathbb{E}[|u_t h_{t,j}(0)|^4]^{1/2} \mathbb{E}[|z_{t,i}|^4]^{1/2} \leq \mathbb{E}[|u_t|^8]^{1/2} \mathbb{E}[|h_{t,j}(0)|^8]^{1/2} \mathbb{E}[|z_{t,i}|^4]^{1/2} < \infty.$$

By Assumption 8, $\mathbb{E}[u_t^2 z_t h_{t,j}(0)^2 z_t']$ is positive definite. It then follows by Theorem 5.25 of White (2001) that $n^{-1/2} \sum_t z_t h_{t,j}(0) u_t$ is asymptotically normal. Thus, $Z'H_j(0)Mu = O_{\mathbb{P}}(n^{1/2})$.

We now consider the statements (*i*)–(*iii*).

(*i*) First, we show that $\theta' Z'H_\kappa(0)Mu = O_{\mathbb{P}}(n^{1/2})$. By the definition of M,

$$Z'H_\kappa(0)Mu = \sum_{t=1}^{n} z_t h_{t,\kappa}(0) u_t - \sum_{t=1}^{n} z_t h_{t,\kappa}(0) z_t' \left(\sum_{t=1}^{n} z_t z_t' \right)^{-1} \sum_{t=1}^{n} z_t u_t. \tag{21}$$

We examine all sums on the right-hand side of (21). First, $h_{t,\kappa}(0)$ is a function of z_t, which implies that, given the moment condition in Assumption 7, $n^{-1} \sum z_t h_{t,\kappa}(0) z_t'$ obeys the ergodic theorem. Second, similarly under Assumptions 1, 2, 3, 7, 8, and $\mathcal{H}_{02}$, $n^{-1} \sum z_t z_t'$ also obeys the ergodic theorem. Third, given the assumptions and the proof of Lemma 1, we have already proved that $n^{-1/2} \sum z_t u_t$ is asymptotically normally distributed. Finally, $n^{-1/2} \sum z_t h_{t,\kappa}(0) u_t$ is asymptotically normal, and the proof is similar to that of the asymptotic normality of $n^{-1/2} \sum_t z_t h_{t,j}(0) u_t$ ($j = \kappa + 1, \ldots, 2\kappa$). All these facts imply that $Z'H_\kappa(0)Mu = O_{\mathbb{P}}(n^{1/2})$.

(*ii*) $n^{-1} G_\kappa' G_\kappa \xrightarrow{\text{a.s.}} A_2$ by the ergodic theorem.

(*iii*) Note that

$$Z'H_\kappa(0)MH_\kappa(0)Z = \sum_{t=1}^{n} z_t h_{t,\kappa}(0)^2 z_t' - \sum_{t=1}^{n} z_t h_{t,\kappa}(0) z_t' \left(\sum_{t=1}^{n} z_t z_t' \right)^{-1} \sum_{t=1}^{n} z_t h_{t,\kappa}(0) z_t'. \tag{22}$$

The limit of (22) is revealed by applying the ergodic theorem to each term on the right-hand side of this expression.

Consequently, $n^{-1}Z'H_\kappa(0)MH_\kappa(0)Z \xrightarrow{\text{a.s.}} \mathbb{E}[g_{t,\kappa}g'_{t,\kappa}]$, where

$$\mathbb{E}[g_{t,\kappa}g'_{t,\kappa}] := \mathbb{E}[z_t H_{2\kappa}(0)^2 z'_t] - \mathbb{E}[z_t H_{2\kappa}(0) z'_t]\mathbb{E}[z_t z'_t]^{-1}\mathbb{E}[z_t H_{2\kappa}(0) z'_t].$$

This completes the proof. ■

Proof of Lemma 7: The distributional equivalence between $\dot{\mathcal{G}}(\cdot)$ and $\ddot{\mathcal{G}}(\cdot)$ can be established by showing that for all γ, $\widetilde{\gamma} \geq 0$, $\mathbb{E}[\ddot{\mathcal{G}}(\gamma)\ddot{\mathcal{G}}(\widetilde{\gamma})] = \mathbb{E}[\dot{\mathcal{G}}(\gamma)\dot{\mathcal{G}}(\widetilde{\gamma})]$. We will proceed in three steps. First, we derive the functional form of $\ddot{\rho}(\gamma, \widetilde{\gamma})$. We show that if $\gamma, \widetilde{\gamma} > 0$, then $\ddot{k}_1(\gamma, \widetilde{\gamma}) = \sum_{n=1}^{\infty} b_n(\gamma)b_n(\widetilde{\gamma})$. This in turn implies that for $\gamma, \widetilde{\gamma} > 0$,

$$\ddot{\rho}(\gamma, \widetilde{\gamma}) = \frac{\sum_{n=1}^{\infty} b_n(\gamma)b_n(\widetilde{\gamma})}{\ddot{k}_1(\gamma, \gamma)^{1/2}\ddot{k}_1(\widetilde{\gamma}, \widetilde{\gamma})^{1/2}}.$$

It follows that the specific functional form of $\ddot{\rho}(\gamma, \widetilde{\gamma})$ can be obtained from this result and (12).

Second, similarly for all $\gamma, \widetilde{\gamma} \geq 0$, we derive the functional form of $\dot{\rho}(\gamma, \widetilde{\gamma})$ and compare it to $\ddot{\rho}(\gamma, \widetilde{\gamma})$. To do all this, we first note that for all $\gamma, \widetilde{\gamma} > 0$,

$$\begin{aligned}
\ddot{k}_1(\gamma, \tilde{\gamma}) &= \frac{1}{4}\mathbb{E}\left[y_{t-1}^2 \tanh\left(\frac{\gamma y_{t-1}}{2}\right)\tanh\left(\frac{\widetilde{\gamma} y_{t-1}}{2}\right)\right] \\
&\qquad - \frac{1}{4}\mathbb{E}\left[y_{t-1}^2 \tanh\left(\frac{\gamma y_{t-1}}{2}\right)\right]\mathbb{E}[y_{t-1}^2]^{-1}\mathbb{E}\left[y_{t-1}^2 \tanh\left(\frac{\widetilde{\gamma} y_{t-1}}{2}\right)\right], \\
&= \frac{1}{4}\mathbb{E}\left[y_{t-1}^2 \tanh\left(\frac{\gamma y_{t-1}}{2}\right)\tanh\left(\frac{\widetilde{\gamma} y_{t-1}}{2}\right)\right]. \qquad (23)
\end{aligned}$$

This follows from that fact that for any $x \in \mathbb{R}$, $\tanh(x) = -\tanh(-x)$ and that y_t follows the Laplace distribution with mean zero and variance 2, so that $\mathbb{E}\left[y_t^2 \tanh\left(\gamma y_t/2\right)\right] = 0$. Given this, we can apply the Dirichlet series to $\tanh(\cdot)$ to obtain the functional form of $\ddot{k}_1(\cdot, \cdot)$. Thus, for any $x \in \mathbb{R}$,

$$\tanh(x) = \operatorname{sgn}(x)\left(1 - 2\sum_{k=0}^{\infty}(-1)^k \exp(-2|x|(k+1))\right)$$

and, furthermore, that

$$\mathbb{E}\left[s_t^2 \exp(-s_t\gamma k)\right] = \frac{2}{(1+\gamma k)^3} \quad \text{and} \quad \mathbb{E}[s_t^2] = 2,$$

where $s_t := |y_t|$ follows the exponential distribution with mean 1 and variance 2. Applying these to (23) yields

$$\begin{aligned}
\ddot{k}_1(\gamma,\widetilde{\gamma}) &= \mathbb{E}\left[\frac{y_t^2}{4}\tanh\left(\frac{\gamma y_t}{2}\right)\tanh\left(\frac{\widetilde{\gamma} y_t}{2}\right)\right] \\
&= \mathbb{E}\left[\frac{s_t^2}{4}\right] - \mathbb{E}\left[\frac{s_t^2}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\exp(-s\gamma k)\right] - \mathbb{E}\left[\frac{s_t^2}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\exp(-s\widetilde{\gamma} k)\right] \\
&\qquad + \mathbb{E}\left[s_t^2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{k+j-2}\exp(-s_t(\gamma k+\widetilde{\gamma} j))\right] \\
&= \frac{1}{2} - \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(1+\gamma k)^3} - \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(1+\widetilde{\gamma} k)^3} + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{j+k-2}\frac{2}{(1+\gamma k+\widetilde{\gamma} j)^3}.
\end{aligned}$$

Next, for $|x|<1$ we have

$$\frac{1}{(1-x)^3} = \sum_{n=1}^{\infty}\frac{n(n+1)}{2}x^{n-1}$$

so that

$$\frac{1}{(1+\gamma k+\widetilde{\gamma} j)^3} = \frac{1}{(1+\gamma k)^3(1+\widetilde{\gamma} j)^3\left(1-\frac{\gamma k}{1+\gamma k}\frac{\widetilde{\gamma} j}{1+\widetilde{\gamma} j}\right)^3},$$

where

$$\frac{1}{\left(1-\frac{\gamma k}{1+\gamma k}\frac{\widetilde{\gamma} j}{1+\widetilde{\gamma} j}\right)^3} = \sum_{n=1}^{\infty}\frac{n(n+1)}{2}\left(\frac{\gamma k}{1+\gamma k}\frac{\widetilde{\gamma} j}{1+\widetilde{\gamma} j}\right)^{n-1}.$$

Therefore,

$$\begin{aligned}
\ddot{k}_1(\gamma,\widetilde{\gamma}) &= \frac{1}{2} - \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(1+\gamma k)^3} - \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(1+\widetilde{\gamma} k)^3} \\
&\qquad + \sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{j+k-2}n(n+1)\frac{(\gamma k)^{n-1}}{(1+\gamma k)^{n+2}}\frac{(\widetilde{\gamma} j)^{n-1}}{(1+\widetilde{\gamma} j)^{n+2}}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{j+k-2}n(n+1)\frac{(\gamma k)^{n-1}}{(1+\gamma k)^{n+2}}\frac{(\widetilde{\gamma} j)^{n-1}}{(1+\widetilde{\gamma} j)^{n+2}} \\
&\quad = 2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(1+\gamma k)^3}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{(1+\widetilde{\gamma} j)^3} + \sum_{n=2}^{\infty}n(n+1)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(\gamma k)^{n-1}}{(1+\gamma k)^{n+2}}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}(\widetilde{\gamma} j)^{n-1}}{(1+\widetilde{\gamma} j)^{n+2}} \\
&\quad = 2a(\gamma)a(\widetilde{\gamma}) + \sum_{n=2}^{\infty}b_n(\gamma)b_n(\widetilde{\gamma}),
\end{aligned}$$

where for $n=2,3,\ldots$,

$$a(\gamma) := \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(1+\gamma k)^3} \quad \text{and} \quad b_n(\gamma) := \sqrt{n(n+1)}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(\gamma k)^{n-1}}{(1+\gamma k)^{n+2}}.$$

In particular,

$$b_1(\gamma) := \frac{1}{\sqrt{2}}(1 - 2a(\gamma)),$$

so that

$$\begin{aligned}\ddot{k}_1(\gamma,\widetilde{\gamma}) &= \frac{1}{2} - a(\gamma) - a(\widetilde{\gamma}) + 2a(\gamma)a(\widetilde{\gamma}) + \sum_{n=2}^{\infty} b_n(\gamma)b_n(\widetilde{\gamma}) \\ &= \frac{1}{2}(1-2a(\gamma))(1-2a(\widetilde{\gamma})) + \sum_{n=2}^{\infty} b_n(\gamma)b_n(\widetilde{\gamma}) \\ &= \sum_{n=1}^{\infty} b_n(\gamma)b_n(\widetilde{\gamma}).\end{aligned}$$

Then, for each $\gamma, \widetilde{\gamma} > 0$,

$$\ddot{\rho}_1(\gamma,\widetilde{\gamma}) := \mathbb{E}[\ddot{\mathcal{G}}_1(\gamma)\ddot{\mathcal{G}}_1(\widetilde{\gamma})] = \frac{\sum_{n=1}^{\infty} b_n(\gamma)b_n(\widetilde{\gamma})}{\ddot{k}_1(\gamma,\gamma)^{1/2}\ddot{k}_1(\widetilde{\gamma},\widetilde{\gamma})^{1/2}}.$$

In addition, for $\gamma > 0$, we examine $\ddot{\rho}_3(\gamma) := \mathbb{E}[\ddot{\mathcal{G}}_1(\gamma)\ddot{\mathcal{G}}_2]$. Note that from (12),

$$\ddot{\rho}_3(\gamma) = \frac{\mathbb{E}[y_t^3 \tanh(\gamma y_t/2)]}{4\sqrt{6}\ddot{k}_1(\gamma,\gamma)^{1/2}} = \frac{1}{32\sqrt{6}\gamma^4\ddot{k}_1(\gamma,\gamma)^{1/2}}\left[48\gamma^4 + P_G\left(3, 1+\frac{1}{2\gamma}\right) - P_G\left(3, \frac{1+\gamma}{2\gamma}\right)\right]$$

as affirmed by Mathematica. It follows that the specific functional form of $\ddot{\rho}(\gamma,\widetilde{\gamma})$ is given as

$$\ddot{\rho}(\gamma,\widetilde{\gamma}) = \begin{cases} \frac{\ddot{k}_1(\gamma,\widetilde{\gamma})}{\ddot{k}_1(\gamma,\gamma)^{1/2}\ddot{k}_1(\widetilde{\gamma},\widetilde{\gamma})^{1/2}}, & \text{if } \gamma > 0 \text{ and } \widetilde{\gamma} > 0; \\ 1, & \text{if } \gamma = 0 \text{ and } \widetilde{\gamma} = 0; \\ \frac{48\gamma^4 + P_G(3,1+\frac{1}{2\gamma}) - P_G(3,\frac{1+\gamma}{2\gamma})}{32\sqrt{6}\gamma^4\ddot{k}_1(\gamma,\gamma)^{1/2}}, & \text{if } \gamma > 0 \text{ and } \widetilde{\gamma} = 0, \end{cases} \quad . \tag{24}$$

Third, we examine the covariance kernel of $\dot{\mathcal{G}}(\cdot)$, viz., $\dot{\rho}(\cdot,\cdot)$. If we let $\gamma, \widetilde{\gamma} > 0$,

$$\dot{\rho}(\gamma,\widetilde{\gamma}) := \mathbb{E}[\dot{\mathcal{G}}(\gamma)\cdot\mathcal{G}(\widetilde{\gamma})] = \frac{\sum_{n=1}^{\infty} b_n(\gamma)b_n(\widetilde{\gamma})}{\ddot{k}_1(\gamma,\gamma)^{1/2}\ddot{k}_1(\widetilde{\gamma},\widetilde{\gamma})^{1/2}} = \frac{\ddot{k}_1(\gamma,\widetilde{\gamma})}{\ddot{k}_1(\gamma,\gamma)^{1/2}\ddot{k}_1(\widetilde{\gamma},\widetilde{\gamma})^{1/2}} = \ddot{\rho}_1(\gamma,\widetilde{\gamma}).$$

Furthermore, by some tedious algebra,

$$\text{plim}_{\gamma\downarrow 0}\ddot{\mathcal{Z}}_1^2(\gamma) = 0, \quad \text{plim}_{\gamma\downarrow 0}\frac{\partial}{\partial\gamma}\ddot{\mathcal{Z}}_1^2(\gamma) = 0, \quad \text{plim}_{\gamma\downarrow 0}\frac{\partial^2}{\partial\gamma^2}\ddot{\mathcal{Z}}_1^2(\gamma) = \frac{1}{8}\{3\sqrt{2}Z_1 + \sqrt{6}Z_2\}^2,$$

$$\text{plim}_{\gamma\downarrow 0}\ddot{k}_1(\gamma,\gamma) = 0, \quad \text{plim}_{\gamma\downarrow 0}\frac{\partial}{\partial\gamma}\ddot{k}_1(\gamma,\gamma) = 0, \quad \text{and} \quad \text{plim}_{\gamma\downarrow 0}\frac{\partial^2}{\partial\gamma^2}\ddot{k}_1(\gamma,\gamma) = 3$$

so that

$$\text{plim}_{\gamma\downarrow 0}\dot{\mathcal{G}}^2(\gamma) = \left(\frac{\sqrt{3}}{2}z_1 + \frac{1}{2}z_2\right)^2,$$

which implies

$$\dot{\mathcal{G}}_2 := \text{plim}_{\gamma\downarrow 0}\dot{\mathcal{G}}(\gamma) = \frac{\sqrt{3}}{2}z_1 + \frac{1}{2}z_2 \sim N(0,1).$$

Consequently, if $\gamma > 0$,

$$\begin{aligned}\mathbb{E}[\dot{\mathcal{G}}(\gamma)\dot{\mathcal{G}}_2] &= \ddot{k}_1(\gamma,\gamma)^{-1/2}\mathbb{E}\left[\ddot{\mathcal{Z}}_1(\gamma)\left(\frac{\sqrt{3}}{2}z_1+\frac{1}{2}z_2\right)\right] = \ddot{k}_1(\gamma,\gamma)^{-1/2}\left[\frac{\sqrt{3}}{2}b_1(\gamma)+\frac{1}{2}b_2(\gamma)\right] \\ &= \frac{1}{32\sqrt{6}\gamma^4\ddot{k}_1(\gamma,\gamma)^{1/2}}\left[48\gamma^4+P_G\left(3,1+\frac{1}{2\gamma}\right)-P_G\left(3,\frac{1+\gamma}{2\gamma}\right)\right]. \qquad (25)\end{aligned}$$

The last equality follows from

$$b_1(\gamma) = \frac{1}{8\sqrt{2}\gamma^3}\left[8\gamma^3 - P_G\left(2,1+\frac{1}{2\gamma}\right)+P_G\left(2,\frac{1+\gamma}{2\gamma}\right)\right],$$

$$b_2(\gamma) = \frac{1}{16\sqrt{6}\gamma^4}\left[6\gamma P_G\left(2,\frac{1}{2\gamma}\right)-6\gamma P_G\left(2,\frac{1+\gamma}{2\gamma}\right)+P_G\left(3,\frac{1}{2\gamma}\right)-P_G\left(3,\frac{1+\gamma}{2\gamma}\right)\right],$$

$$P_G\left(2,\frac{1}{2\gamma}\right)-P_G\left(2,1+\frac{1}{2\gamma}\right) = -16\gamma^3, \quad \text{and} \quad P_G\left(3,\frac{1}{2\gamma}\right)-P_G\left(3,1+\frac{1}{2\gamma}\right) = 96\gamma^4,$$

as obtained by Mathematica. Equation (25) then leads to the following functional form for $\dot{\rho}(\gamma,\widetilde{\gamma}) := \mathbb{E}[\dot{\mathcal{G}}(\gamma)\dot{\mathcal{G}}(\widetilde{\gamma})]$:

$$\dot{\rho}(\gamma,\widetilde{\gamma}) = \begin{cases} \frac{\ddot{k}_1(\gamma,\widetilde{\gamma})}{\ddot{k}_1(\gamma,\gamma)^{1/2}\ddot{k}_1(\widetilde{\gamma},\widetilde{\gamma})^{1/2}}, & \text{if } \gamma>0 \text{ and } \widetilde{\gamma}>0; \\ 1, & \text{if } \gamma=0 \text{ and } \widetilde{\gamma}=0; \\ \frac{48\gamma^4+P_G(3,1+\frac{1}{2\gamma})-P_G(3,\frac{1+\gamma}{2\gamma})}{32\sqrt{6}\gamma^4\ddot{k}_1(\gamma,\gamma)^{1/2}}, & \text{if } \gamma>0 \text{ and } \widetilde{\gamma}=0, \end{cases}$$

which is identical to the functional form of $\ddot{\rho}(\cdot,\cdot)$ in (24). This allows the conclusion that $\ddot{\mathcal{G}}(\cdot)$ has the same distribution as $\dot{\mathcal{G}}(\cdot)$. ■

In the following, we provide additional supplementary claim in (13) that is given in the following lemma:

Lemma 8. *Given the DGP and Model conditions in Section 3.2,*

$$\lim_{\widetilde{\gamma}\downarrow 0}\ddot{\rho}_1(\gamma,\widetilde{\gamma})^2 = \left(\frac{1}{32\sqrt{6}\gamma^4\ddot{k}_1(\gamma,\gamma)^{1/2}}\left[48\gamma^4+P_G\left(3,1+\frac{1}{2\gamma}\right)-P_G\left(3,\frac{1+\gamma}{2\gamma}\right)\right]\right)^2. \qquad \square$$

Lemma 8 implies that $\text{plim}_{\gamma\downarrow 0}\ddot{\mathcal{G}}_1(\gamma)^2 = \ddot{\mathcal{G}}_2^2$, so that $\sup_{\gamma\in\Gamma}\ddot{\mathcal{G}}_1(\gamma)^2 \geq \ddot{\mathcal{G}}_2^2$ and $QLR_n \Rightarrow \sup_{\gamma\in\Gamma}\ddot{\mathcal{G}}_1(\gamma)^2$.

Proof of Lemma 8: From the definition of $\ddot{\rho}_1(\gamma,\widetilde{\gamma})$, we note that

$$\ddot{\rho}_1(\gamma,\widetilde{\gamma})^2 := \frac{\ddot{k}_1(\gamma,\widetilde{\gamma})^2}{\ddot{k}_1(\gamma,\gamma)\ddot{k}_1(\widetilde{\gamma},\widetilde{\gamma})}.$$

Furthermore, we have

$$\text{plim}_{\widetilde{\gamma}\downarrow 0}\ddot{k}_1(\gamma,\widetilde{\gamma})^2 = 0, \quad \text{plim}_{\widetilde{\gamma}\downarrow 0}\frac{\partial}{\partial\widetilde{\gamma}}\ddot{k}_1(\gamma,\widetilde{\gamma})^2 = 0,$$

$$\text{plim}_{\widetilde{\gamma}\downarrow 0}\frac{\partial^2}{\partial\widetilde{\gamma}^2}\ddot{k}_1(\gamma,\widetilde{\gamma})^2 = \left(\frac{48\gamma^4+P_G(3,1+\frac{1}{2\gamma})-P_G(3,\frac{1+\gamma}{2\gamma})}{32\sqrt{2}\gamma^4}\right)^2,$$

$$\text{plim}_{\widetilde{\gamma}\downarrow 0}\ddot{k}_1(\widetilde{\gamma},\widetilde{\gamma})=0, \quad \text{plim}_{\widetilde{\gamma}\downarrow 0}\frac{\partial}{\partial\widetilde{\gamma}}\ddot{k}_1(\widetilde{\gamma},\widetilde{\gamma})=0, \quad \text{and} \quad \text{plim}_{\widetilde{\gamma}\downarrow 0}\frac{\partial^2}{\partial\widetilde{\gamma}^2}\ddot{k}_1(\widetilde{\gamma},\widetilde{\gamma})=3$$

by some tedious algebra using Mathematica. This property implies that

$$\begin{aligned}\lim_{\widetilde{\gamma}\downarrow 0}\ddot{\rho}_1(\gamma,\widetilde{\gamma})^2 &= \frac{1}{3\ddot{k}_1(\gamma,\gamma)}\left(\frac{48\gamma^4+P_G(3,1+\frac{1}{2\gamma})-P_G(3,\frac{1+\gamma}{2\gamma})}{32\sqrt{2}\gamma^4}\right)^2 \\ &= \left(\frac{1}{32\sqrt{6}\gamma^4\ddot{k}_1(\gamma,\gamma)^{1/2}}\left[48\gamma^4+P_G\left(3,1+\frac{1}{2\gamma}\right)-P_G\left(3,\frac{1+\gamma}{2\gamma}\right)\right]\right)^2.\end{aligned}$$

This completes the proof. ■

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Transition Variable	$LM_{1,n}$	$LM_{2,n}$	$LM_{3,n}$	$LM_{4,n}$	QLR_n^L	QLR_n^E
Δg_{t-1}	$0.752^{\dagger}$	0.788	0.447	0.698	**0.033**	**0.001**
Δg_{t-2}	$0.192^{\dagger}$	0.349	0.394	0.286	0.125	**0.024**
Δr_t	$0.076^{\dagger}$	0.087	0.625	0.448	0.400	**0.000**
Δr_{t-1}	$0.904^{\dagger}$	0.819	0.845	0.766	0.979	0.068
Δp_t	$\mathbf{0.032}^{\dagger}$	**0.016**	**0.039**	0.115	**0.028**	0.474

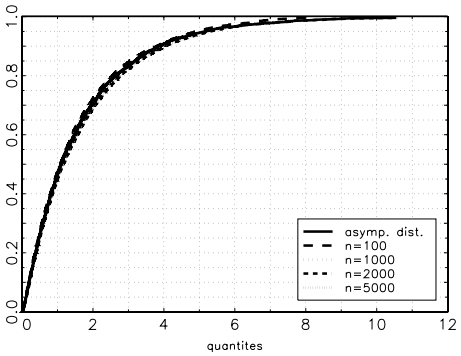
Table 1: LINEARITY TESTS FOR THE INDUSTRIAL PRODUCTION. Notes: The p-values of the linearity tests for the German money demand function are provided. The p-values are obtained using the observations from 1960Q1 to 1996Q4, and the p-values in the second panel are obtained using the observations from 1961Q1 to 2015Q1. The data are obtained from the data archive of the *Journal of Applied Econometrics* (see footnote 1). The p-values attached by the superscript '†' correspond to the p-values computed by Lütkepohl, Teräsvirta, and Wolters (1999). Boldface p-values indicate significance levels less than or equal to 0.05.

Periods	Transition Variable	$LM_{1,n}$	$LM_{2,n}$	$LM_{3,n}$	$LM_{4,n}$	QLR_n^L	QLR_n^E
1968.06∼1999.12	$\Delta_{12} y_{t-1}$	0.150	0.532	0.412	0.895	**0.015**	0.086
	$\Delta_{12} y_{t-2}$	**0.037**	0.093	0.057	0.195	**0.002**	0.092
	$\Delta_{12} y_{t-3}$	0.162	0.326	0.163	0.555	**0.050**	0.164
	$\Delta_{12} y_{t-4}$	0.665	0.745	0.546	0.619	**0.032**	0.309
	$\Delta_{12} y_{t-5}$	0.662	0.886	0.954	0.830	**0.016**	0.203
	$\Delta_{12} y_{t-6}$	0.588	0.306	0.121	0.234	**0.006**	0.357
1968.06∼2015.08	$\Delta_{12} y_{t-1}$	**0.000**	**0.000**	**0.000**	0.098	**0.000**	**0.003**
	$\Delta_{12} y_{t-2}$	**0.000**	**0.000**	**0.000**	**0.016**	**0.000**	**0.004**
	$\Delta_{12} y_{t-3}$	**0.001**	**0.000**	**0.008**	**0.045**	**0.000**	**0.020**
	$\Delta_{12} y_{t-4}$	**0.008**	**0.012**	0.070	0.111	**0.002**	0.154
	$\Delta_{12} y_{t-5}$	**0.038**	0.237	0.274	0.861	**0.003**	0.264
	$\Delta_{12} y_{t-6}$	**0.003**	0.068	**0.017**	0.582	**0.000**	0.823

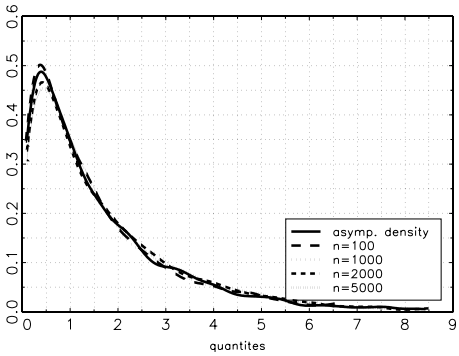
Table 2: LINEARITY TESTS FOR THE MONTHLY US UNEMPLOYMENT RATE. Notes: The p-values of the linearity tests for the first differenced monthly US unemployment rate are provided. The p-values in the top panel are obtained using observations from 1968.06 to 1999.12, and the p-values of the bottom panel are obtained using observations from 1968.06 to 2015.08. The null linear model is given as AR(15) by AIC, and the twelve-month differences are considered as a transition variable. Boldface p-values indicate significance levels less than or equal to 0.05.

Periods	d	$LM_{1,n}$	$LM_{2,n}$	$LM_{3,n}$	$LM_{4,n}$	QLR_n^L	QLR_n^E
1961Q1∼1986Q1	1	0.393	0.716	0.851	0.832	0.565	0.262
	2	0.129	0.209	0.389	0.377	0.121	0.053
	3	0.375	0.362	0.710	0.750	**0.045**	0.189
	4	0.233	0.184	0.091	0.192	0.275	0.247
	5	0.053	0.275	0.310	0.109	0.183	0.548
	6	**0.015**	**0.0178**	0.063	0.390	0.207	0.229
	7	0.151	0.226	0.407	0.496	0.077	0.613
	8	0.550	0.182	0.067	0.221	0.070	0.696
	9	0.298	0.272	0.077	0.526	0.345	0.865
1961Q1∼2015Q1	1	**0.000**	**0.000**	**0.000**	**0.000**	**0.016**	0.148
	2	**0.000**	**0.000**	**0.000**	**0.000**	0.057	**0.020**
	3	**0.024**	0.060	0.061	0.443	**0.000**	0.717
	4	0.140	**0.020**	**0.044**	**0.048**	**0.033**	**0.050**
	5	0.604	0.248	0.141	0.411	0.207	0.738
	6	0.158	0.264	0.498	0.503	0.168	0.190

Table 3: LINEARITY TESTS FOR THE GERMAN INDUSTRIAL PRODUCTION. Notes: The p-values of the linearity tests for the differenced log of the quarterly German industrial production. The p-values in the top panel are obtained using the observations from 1961Q1-1986Q1, and the p-values in the bottom panel are obtained using the observations from 1961Q1 to 2015Q1. In particular, a dummy variable is multiplied to the lagged dependent variables to accommodate the Germany reunification effect to the monetary policy. The data source is OECD Main Economic Indicators. AR(9) is selected by AIC for the first data. On the other hand, AIC selects 6 lagged dependent variables and 2 lagged dependent variables multiplied by the dummy variable for the second data. Boldface p-values indicate significance levels less than or equal to 0.05.



$\Gamma_1 = [0, 2]$



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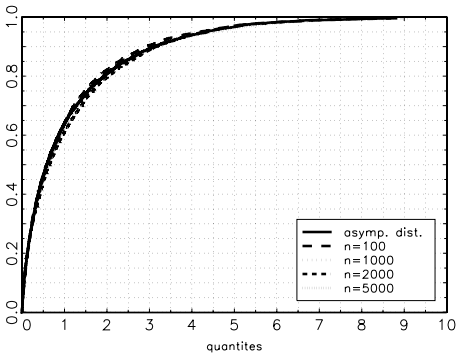

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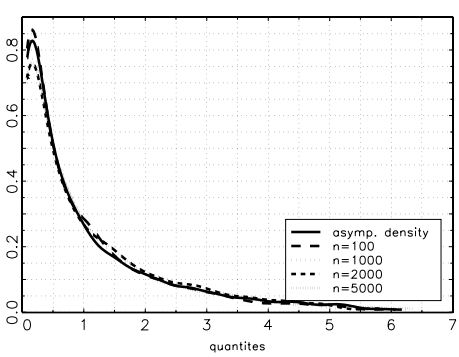

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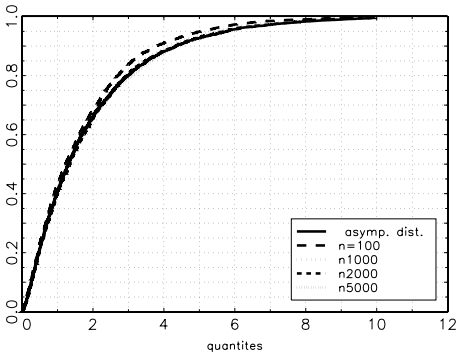

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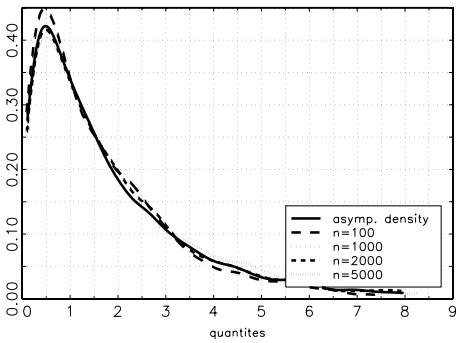

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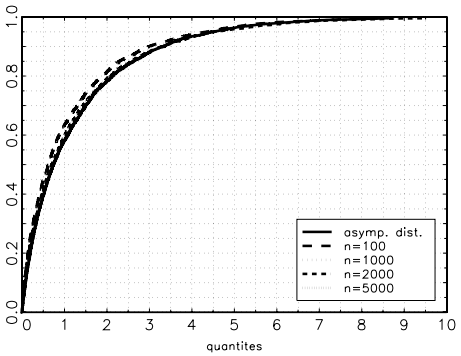

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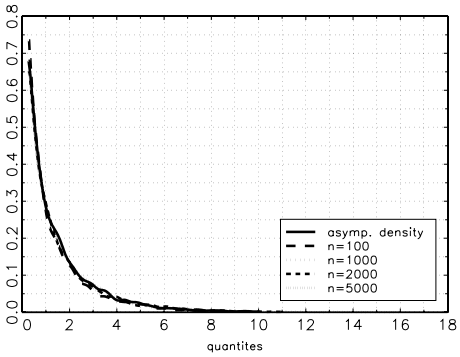
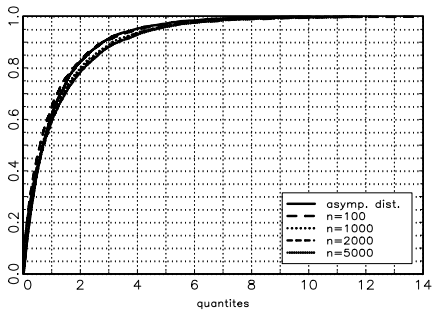


$\Gamma_4 = [0, 5]$

Figure 1: EMPIRICAL NULL DISTRIBUTIONS OF THE QLR STATISTIC AND ITS NULL LIMIT DISTRIBUTION (ESTAR MODEL CASE). Notes: (*i*) Number of Iterations: 5,000; (*ii*) DGP: $y_t = 0.5y_{t-1} + u_t$ and $u_t \sim$ IID $N(0, 1)$; (*iii*) Model: $y_t = \pi y_{t-1} + \theta y_{t-1}\{1 - \exp(-\gamma y_{t-1}^2)\} + u_t$ and $u_t \sim$ IID $N(0, 1)$; and (*iv*) $\Gamma_1 = [0, 2]$, $\Gamma_2 = [0, 3]$, $\Gamma_3 = [0, 4]$, $\Gamma_4 = [0, 5]$.



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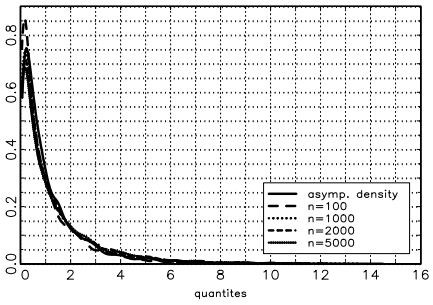

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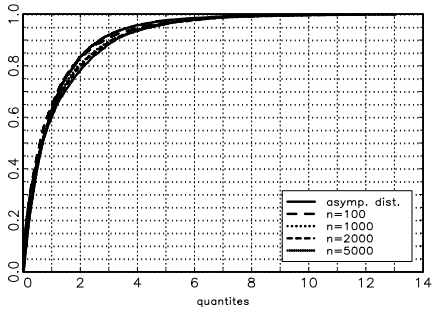

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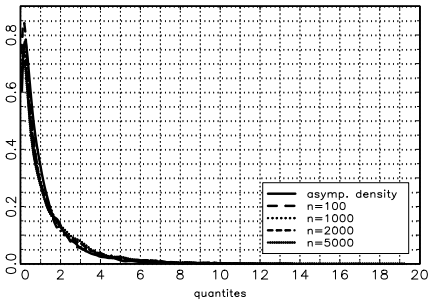

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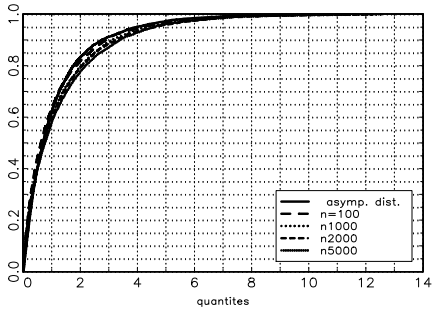
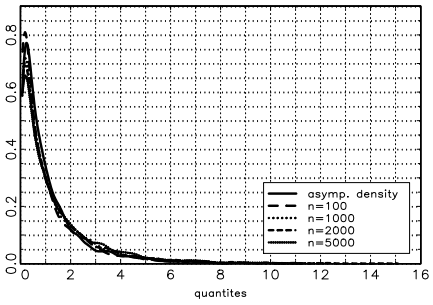
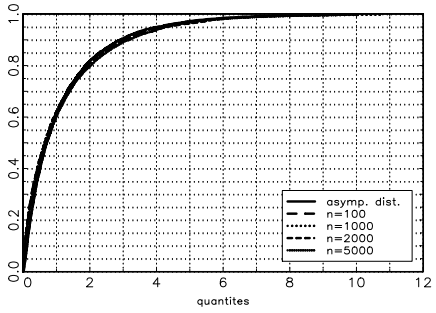


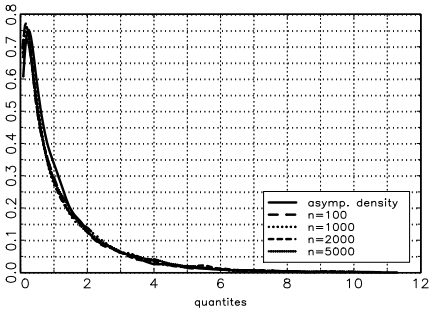
$\Gamma_3 = [0, 4]$



$\Gamma_3 = [0, 4]$

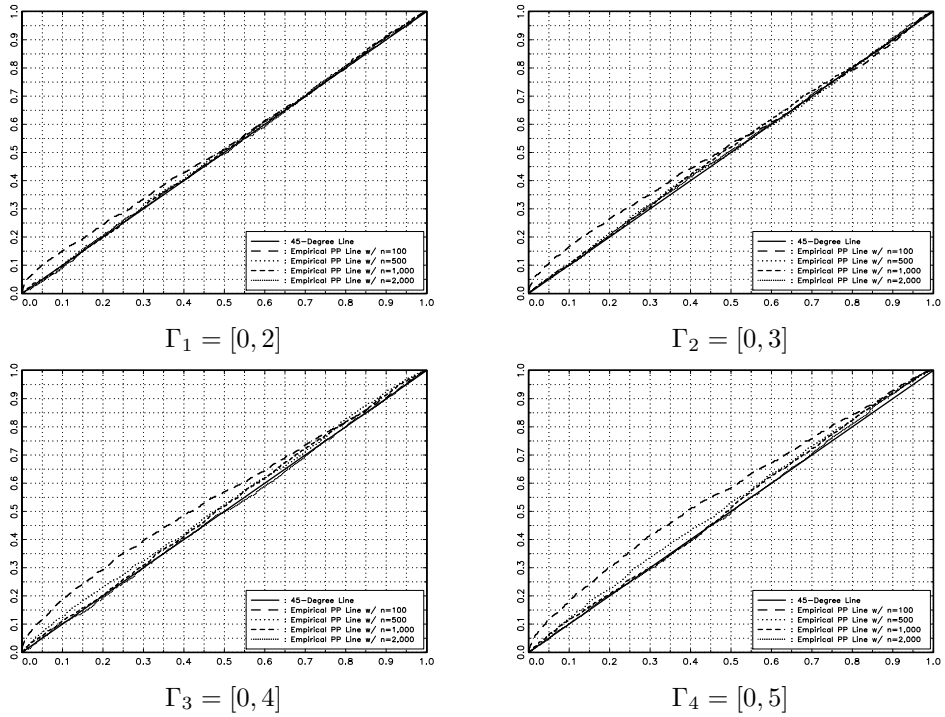


$\Gamma_4 = [0, 5]$



$\Gamma_4 = [0, 5]$

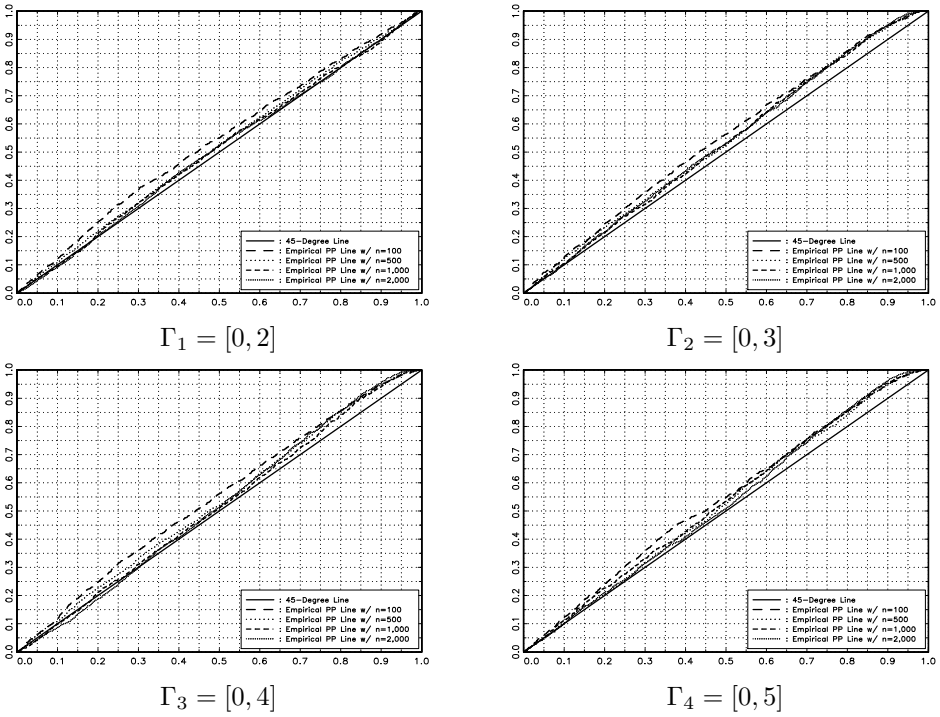
Figure 2: EMPIRICAL NULL DISTRIBUTIONS OF THE QLR STATISTIC AND ITS NULL LIMIT DISTRIBUTION (LSTAR MODEL CASE). Notes: (*i*) Number of Iterations: 5,000; (*ii*) DGP: $y_t = 0.5y_{t-1} + u_t$ and $u_t = i_t\ell_t$, where $\{i_t\}$ is an IID sequence in which $\mathbf{P}\{i_t = 1\} = 1 - 0.5^2$ and $\{\ell_t\} \sim$ Laplace$(0, 2)$; (*iii*) Model: $y_t = \pi y_{t-1} + \theta y_{t-1}\{(1 + \exp(-\gamma y_{t-1}))^{-1}\} + u_t$ and $u_t = i_t\ell_t$, where $\{i_t\}$ is an IID sequence in which $\mathbf{P}\{i_t = 1\} = 1 - 0.5^2$ and $\{\ell_t\} \sim$ Laplace$(0, 2)$; and (*iv*) $\Gamma_1 = [0, 2]$, $\Gamma_2 = [0, 3]$, $\Gamma_3 = [0, 4]$, $\Gamma_4 = [0, 5]$.



$\Gamma_1 = [0, 2]$ $\Gamma_2 = [0, 3]$

$\Gamma_3 = [0, 4]$ $\Gamma_4 = [0, 5]$

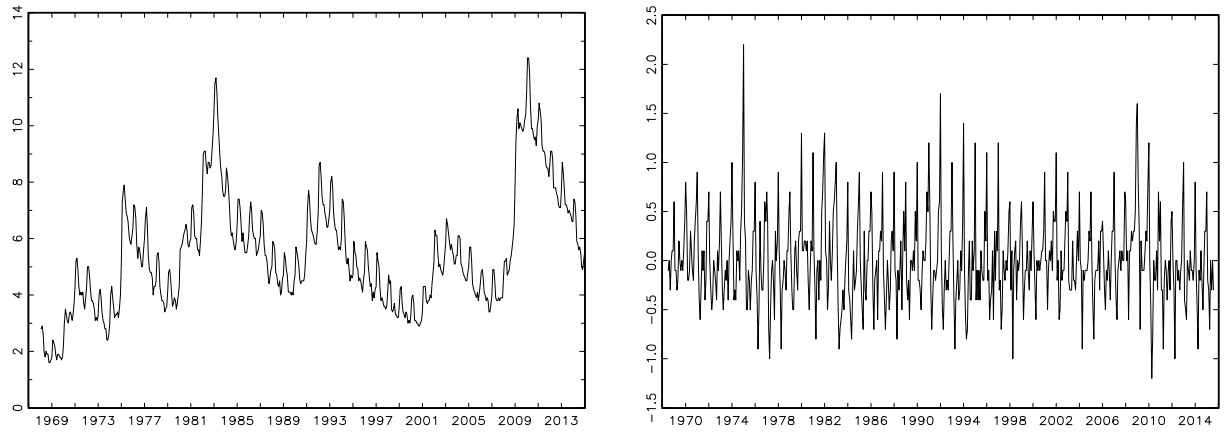
Figure 3: PP PLOTS OF THE QLR STATISTIC USING THE WEIGHTED BOOTSTRAP (ESTAR MODEL CASE). Notes: (*i*) Number of Iterations: 2,000, Bootstrap Iterations: 300; (*ii*) DGP: $y_t = 0.5y_{t-1} + u_t$ and $u_t \sim$ IID $N(0, 1)$; (*iii*) Model: $y_t = \pi y_{t-1} + \theta y_{t-1}\{1 - \exp(-\gamma y_{t-1}^2)\}u_t$ and $u_t \sim$ IID $N(0, 1)$; and (*iv*) $\Gamma_1 = [0, 2]$, $\Gamma_2 = [0, 3]$, $\Gamma_3 = [0, 4]$, $\Gamma_4 = [0, 5]$.



$\Gamma_1 = [0, 2]$ $\Gamma_2 = [0, 3]$

$\Gamma_3 = [0, 4]$ $\Gamma_4 = [0, 5]$

Figure 4: PP PLOTS OF THE QLR STATISTIC USING THE WEIGHTED BOOTSTRAP (LSTAR MODEL CASE). Notes: (*i*) Number of Iterations: 2,000, Bootstrap Iterations: 300; (*ii*) DGP: $y_t = 0.5y_{t-1} + u_t$ and $u_t = i_t\ell_t$, where $\{i_t\}$ is an IID sequence in which $\mathbf{P}\{i_t = 1\} = 1 - 0.25$ and $\{\ell_t\} \sim$ Laplace$(0, 2)$; (*iii*) Model: $y_t = \pi y_{t-1} + \theta y_{t-1}\{(1+\exp(-\gamma y_{t-1}))^{-1} - 1/2\} + u_t$ and $u_t = i_t\ell_t$, where $\{i_t\}$ is an IID sequence in which $\mathbf{P}\{i_t = 1\} = 1-0.5^2$ and $\{\ell_t\} \sim$ Laplace$(0, 2)$; and (*iv*) $\Gamma_1 = [0, 2]$, $\Gamma_2 = [0, 3]$, $\Gamma_3 = [0, 4]$, $\Gamma_4 = [0, 5]$.



The US Unemployment Rate Differenced US Unemployment Rate

Figure 5: THE US UNEMPLOYMENT RATE, 1968.01-2015.08

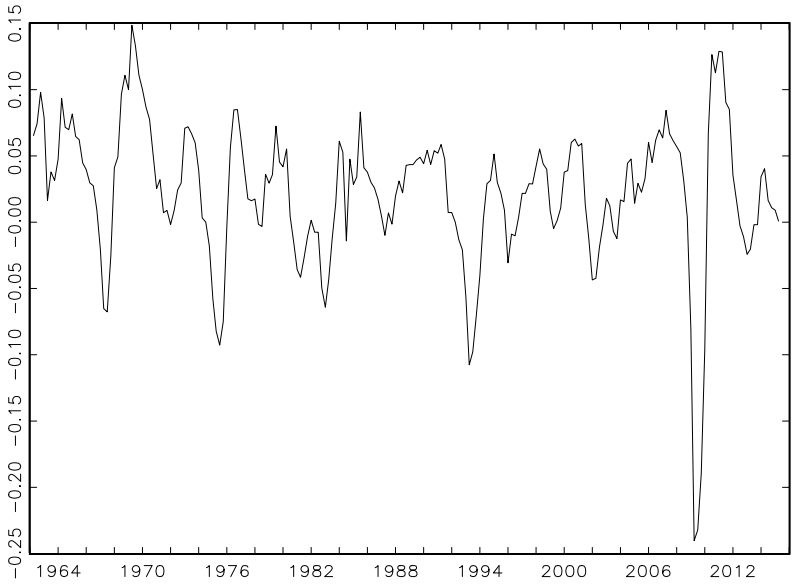


Figure 6: DIFFERENCED LOG OF THE QUARTERLY INDUSTRIAL PRODUCTION IN GERMANY, 1961Q1-2014Q4.